The Vibrating String In Keyboard Instruments

Thesis

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UNRESTRICTED

THE VIBRATING STRING IN KEYBOARD INSTRUMENTS

Submitted to

THE OPEN UNIVERSITY

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IN CANDITURE FOR THE DEGREE OF

MASTER OF PHILOSOPHY

BY

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The vibrating string in keyboard instruments J. Lincoln

Abstract

In this thesis some aspects of the acoustics of the piano and harpsichord are discussed.

Initially the wave equation for the vibrating string is derived. Since the strings in the piano are not perfectly flexible the wave equation is extended to take into account the stiffness of the strings.

The wave equation thus modified affects the frequencies of the partials of the strings. The predicted frequencies are compared with the actual frequencies on a piano. The discussion is then continued to show how this affects the actual tuning of the instrument. Computer and programmable calculator programmes are used in these predictions.

The theoretical basis for the spectra of piano and harpsichord strings are discussed and brief notes included on the effects of hammer width and string stiffness.

The final experimental section deals with the reverberation times for piano and harpsichord strings, taking account of the effects of multiple stringing. Multiple stringing is found to affect the piano strings but not the harpsichord strings reverberation times. No accurate measurements of harpsichord reverberation times could be found in the literature, and the experiments were extended to include the effects of the buff stop.

The conclusions contain several suggestions for further study and stress the importance of piano scaling. It is noted that scaling is also of importance to the musician when selecting instruments for the performance of duets, especially with regard to tuning compatibility.

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6 Appendices

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Thanks are also due to Miss Julie Walker who checked and typed the bibliography, Mr John Spice who answered many practical problems connected with piano tuning, and Mr M Kemp for his advice on the use of the computer.

List of symbols

Listed below are the symbols used in the following paper with their meaning and page where they are first used or defined.

Symbol	Meaning	Page
A	Amplitude	1.5
B	Amplitude Inharmonicity factor	1.5 2.8
c	Velocity of propagation of a wave	1.5
e	Base of natural logarithms	
E	Young's modulus of elasticity Energy of a vibrating string	2.4 1.12
f	Frequency	1.7
F	Restoring force Idealised fundamental frequency	2.4 2.8
h	Deformation of a plucked string	4.2
j	Imaginary unit $\sqrt{-1}$	
k	Wavelength constant or wave number	1.5
K	Complex amplitude reflection coefficient	4.9
L	Length of string	1.7
M	Total mass of string	1.12
n	Mode number	1.7
r	Radius of string	1.9
S	Cross section area of string	2.4
t	Time	1.3
T	Tension Period of vibration	1.2 1.8
T	Contact time of piano hammer	4.8
T_{60}	Reverberation time	4.11
u	(v) x (infinitely small length of string)	4.4
v	velocity of string	1.12

Symbol	Meaning	Page
V	Velocity amplitude of a particle of a string	1.13
W_h	Hammer width	4.7
x	Rectangular co-ordinate	
y	Rectangular co-ordinate	
Y	Complex Young's Modulus	4.22
Z_1	Input impedance	1.10
Z_S	Characteristic impedance	1.9
Z_T	Termination impedance	4.9
α	Angle	1.3
α_T	Power transmission coefficient	4.9
α_R	Power reflection coefficient	4.9
β	String stiffness constant	2.5
δ_n	Inharmonicity of n^{th} mode in cents	2.8
$\mathcal{E}$	Space average energy density	1.13
θ	Angle	1.3
κ	Radius of gyration	2.3
λ	Wavelength	1.5
μ	Linear density	1.3
π	Ratio of circumference to diameter (circle)	
ρ	Density	1.9
ω	Angular frequency $= 2\pi f$	1.5

ADDENDA AND CORRIGENDA

Page 1.7, line 18.

should read 'where f_n is the frequency of the n^{th} mode'

Page 1.9, Note on section 1.5.1

In theoretical treatments of the vibrating string, 'r' is often used to represent the radius of the string. This has been followed here. However, in practice, it is the diameter of the string which is usually measured using a micrometer. Hence in practical discussions such as Appendix 1 the diameter 'd' of the string is a more appropriate measurement.

The characteristic impedance of a string is often denoted by Z_c rather than Z_s as in this thesis.

Page 1.9, line 26.

for 'independant' read 'independent'

Page 1.10, Note on section 1.5.2.

This section deals with a finite string terminated at x=0. At x=0 the string is driven by some external agent, hence the derivation is of the driving point impedance , being the ratio of driving force to the velocity of the end point of the string.

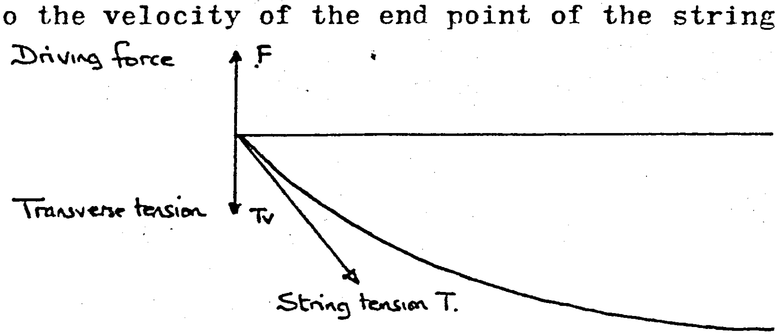


Fig 0.1

At $x=0$

$$T_v = T\left(\frac{\partial y}{\partial x}\right)_{x=0}$$

$$F = -T\left(\frac{\partial x}{\partial x}\right)_{x=0}$$

Hence, the graphs on page 1.11, fig.1.3 represent driving point impedance and the corresponding r.m.s. velocity and r.m.s. driving force. Note that since the quantities are r.m.s. values, the graphs do not indicate the situation at a given instant of time.

If the input impedance is defined as the ratio of transverse tension to velocity at the end point of the string, the resulting graph of input impedance would be a cot curve and not the -cot curve shown.

<u>Page 1.12, lines 14 and 15.</u>

should read 'the maximum potential energy must have the same value as the maximum kinetic energy'

<u>Page 1.13, from line 13.</u>

should read 'Since the amplitude of the standing wave is twice that of the components, then from eqn. 1.17

$$\text{Energy density} = \mathcal{E} = \mu \omega^2 A^2 \qquad (1.19)$$

where A is half the amplitude of the standing wave.

<u>Page 1.14, line 7.</u>

The equation should be labelled ' (Eqn. 1.21) '

<u>Page 1.15, line 21.</u>

for 'this if inadequate' read 'this is inadequate'.

<u>Page 1.16, Note on figure 1.4</u>

This diagram shows the motion of the bridge of a piano. For clarification of the construction of the bridge refer to fig 0.2 on page 0.11.

Returning to fig. 1.4, the upper diagram shows the relationship between the motion of the string and the motion of the bridge when the bridge is 'spring' controlled. The effect is to lower the frequency of the particular mode of the string being considered. The lower diagram of fig 1.4 shows a mass controlled bridge, the effectbeing to raise the mode frequency of the string.

Figure 0.2

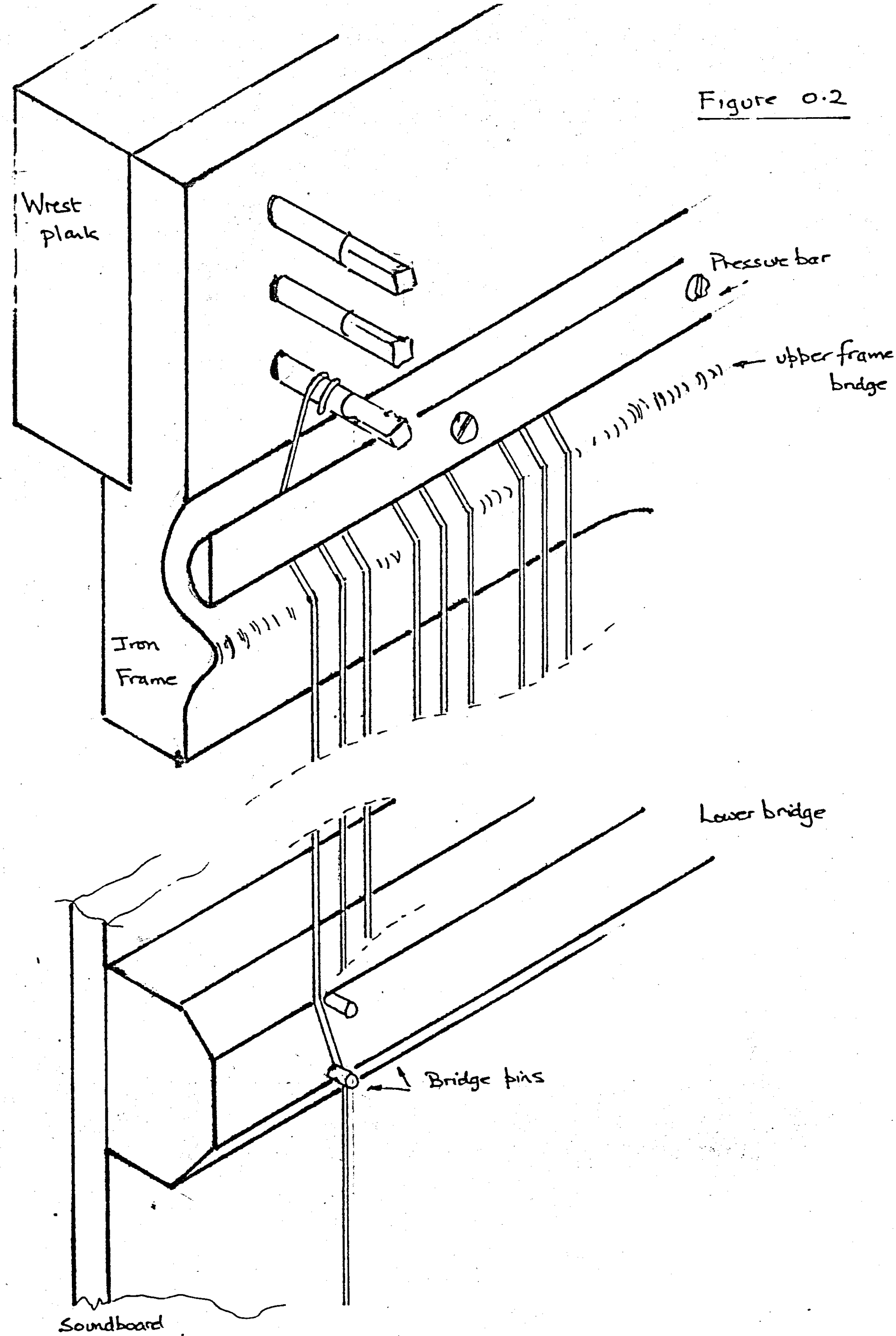
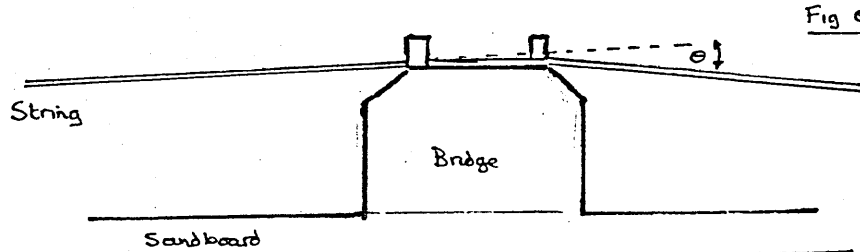


Fig 0.3



Page2.2, line 14

should read ' $M \propto 1/R$ '

Page 2.5, line 10

equation 2.6 should read

$$k^4 - \frac{k^2}{4\beta^2 L^2} - \frac{f^2}{16\beta^2 L^4 f_o^2} = 0$$

Page 2.7, line 19

equation should read

$$k_2 = \frac{n}{2L(1+\epsilon)}$$

Page 2.8, Note on equation 2.17

In this equation the β is a constant obtained from experiment and therefore is not identical with the β values referred to earlier.

Page 2.8, Note on equation 2.19

An expansion and approximation of ln(1 + x) has been used to obtain this equation, and also the approximation $F = f_o$

Page 2.11, line 22

for 'infromation' read 'information'

Page 2.15, line 4

equation 2.26 should read $F = f_{nom} \times 2^{\frac{\delta_o}{1200}}$

Page 2.23, line 14

for 'uproght' read 'upright'

Page 2.23, Note on section 2.7.3

An abrupt change in inharmonicity would be audible as a distinct change in tone and hence undesirable. Occasionally a string exhibits a high value of inharmonicity due to a twist or having been kinked and later straightened; - this string would be referred to as 'false' by a tuner, and would cause difficulties in tuning the instrument.

Page 3.8, line 3

should read 'on the values of β which in turn is dependent on the string '

Page 3.14, Note on final paragraph

The graph (after Railsbeck) shown by Schick and Young shows the stretch in tuning over the whole range of the piano and not just the tuning octave.

Page 4.6, line 11

for 'supperposition' read superposition'

Page 4.6, line 20

Since λ_1 is not used in the quoted formulae, this line should be omitted.

Page 4.7, line 11

for 'recieve' read 'receive'

Page 4.10, line 12

for 'eqn. 1.6 may' read 'eqn. 1.16 may'

Page 4.11, line 10

for 'give' read 'gives'

Page 4.11, Note on section 4.5.1

Triple stringing is shown in figure 0.2 on page 0.11 . This figure also shows that at the upper bridge the sounding length of the string is terminated in a 'hinge' type termination over the upper iron frame. At the lower end of the sounding length of the string it is effectively clamped in two directions; by the offset or side-draught caused by the bridge pins, and by the change in angle (down bearing angle θ) made by the string as it passes over the bridge, shown in fig. 0.3 .

Comments on the coupled motion of piano strings

Since writing the following thesis, the work of G. Weinreich has been brought to my attention. The following papers have been consulted in the compilation of this addenda.

WEINREICH,G., "Coupled piano strings" J.A.S.A., vol 62, no 6, Dec 1977, pp. 1474-1484.

WEINREICH,G., "The coupled motions of piano strings" Scientific American, Jan 1977, pp.118-127.

The instrument used by Weinreich in his experiments was a Steinway grand piano. The use of a grand piano leads to the following convention which is used in the following discussion - string motion in the plane of the striking hammer (perpendicular to the soundboard) is referred to as vertical motion ; - and motion at right angles to the plane of the striking hammer (parallel to the soundboard) is referred to as horizontal motion.

Weinreich shows that a single string can exhibit a double decay. This is due to the existence of two polarisations of vibrations of the string. The vertical mode of oscillation is the main motion excited by the hammer, but irregularities in the hammer may excite some horizontal motion. The vertical mode of oscillation decays much faster than the horizontal mode since the resistive part of bridge admittance is much greater in the vertical direction. The reactive part of bridge admittance is approximately the same in both directions.

When two strings cross the bridge of a piano at almost the same place the decay rate will depend upon the phase relationship between the vibration of the two strings. At one extreme of phase relationship they may vibrate in anti-phase, in this case the bridge will experience no net force, and the decay rate will be zero since the only mechanism being considered for dissipation is that of bridge motion. At the other extreme of phase relationships the strings may vibrate in phase, doubling the motion of the bridge and doubling the decay rate from the single string case. A similar effect will be observed for triple stringing, although the situation could be very complex due to phase relationships.

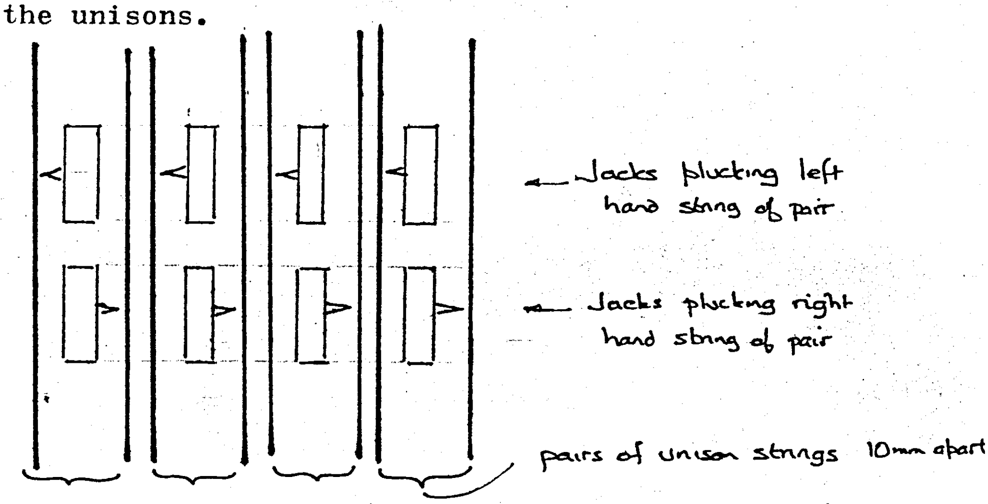
In practice the motion of the strings will neither be exactly the same (symmetric) or exactly opposed (anti-symmetric), and minor imperfections in the hammer could result in the amplitudes of the string vibrations not being equal. If the two strings are in phase, but one has a larger amplitude than the other, both will lose energy at a faster rate than if it were vibrating alone due to the assistance given by the other string to bridge motion. The string with the lesser initial amplitude will decay through zero and vibrate with a negative amplitude, i.e. in anti-phase. The two amplitudes will asymptotically approach one another with opposite phases until they are exactly anti-symmetric. It is the initial symmetric motion of the strings that give rise to the initial sound and the later anti-symmetric motion of the strings which give rise to the aftersound.

If the two strings begin their motion anti-symmetrically and are not at exactly the same frequency, the string with the higher natural frequency will advance in phase so that the strings are no longer purely anti-symmetric. This will exert a small force on the bridge which is approximately a quarter of a cycle out of phase with both of the strings (leading one string and lagging the other). The bridge will begin to move in phase with this force if the impedance of the bridge is purely resistive. For the in-phase string the bridge is acting as a spring controlled support and for the anti-phase string the bridge is acting as a mass controlled support. Referring to fig 1.4 and the note on page 0.10 it can be seen that the frequency of the in-phase string will be raised and the frequency of the anti-phase string lowered, until they are both vibrating at the same frequency. Hence a slight mis-tuning will result in a sound of single frequency with a slow decay. A similar result is obtained if the two strings begin their vibration symmetrically. However, if the mis-tuning is larger, then the resultant sound will beat at the difference frequency, and the decay rate of the symmetric and anti-symmetric motion will become equal to the decay rate for uncoupled strings.

The double decay is due to both the mistuning and the contributionsdue to the horizontal polarisation and the anti-symmetric motion. The tuner can control the decay in some measure by the way in which he tunes the unisons.

Fig 0.4

Top view of arrangement of harpsichord strings and jacks.



The lack of double decay in the harpsichord is probably due to the lack of coupling between unison strings. From fig. 0.4 it can be seen that due to the arrangement of the jacks, the separation of the unison strings is in the order of 10mm. For the piano, the separation is in the order of 3-4mm, and the bridge structure is also more substantial. Hence the coupling between unison strings of a harpsichord is less than that of the piano.

Note on inharmonicity experiments

The data from the inharmonicity experiments in section 2.7.2 was analysed using a statistical programme on a pocket calculator. Graphs of this data were also plotted, but not included in the thesis. A typical graph (showing the data from page 2.17) is shown overleaf on page 0.18 . Graphs of each of the tables of results from measurements of inharmonicity are similar. They each show very good correlation to the straight line for the higher harmonics, and apparently random results for the lower harmonics. In the region of these lower harmonics the bandwidth of the analyser is of the same order as the amount of inharmonicity.

Example;

3rd partial of the C2 string, measured frequency 197 Hz

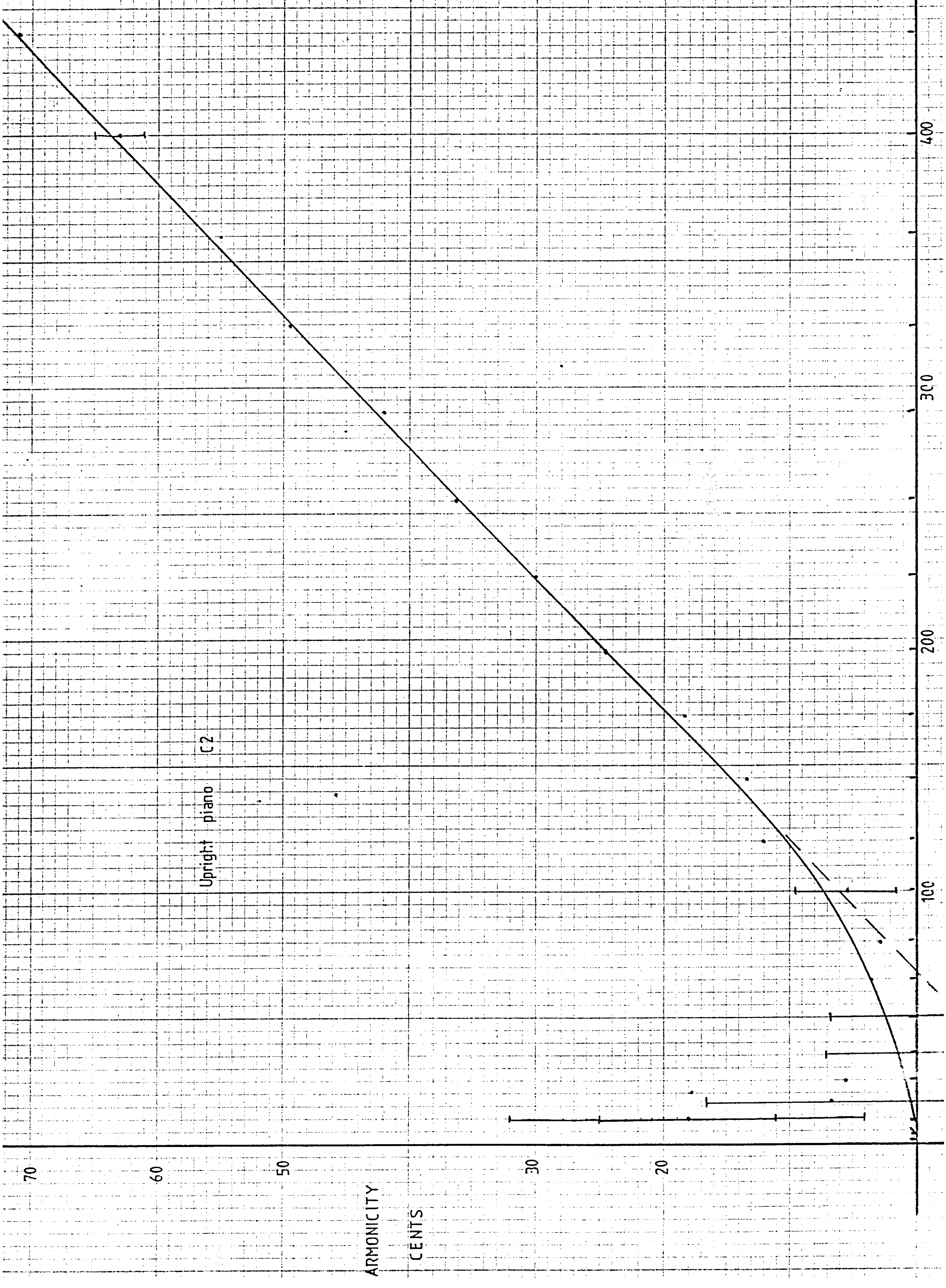
Harmonic frequency = 3 x 64.975 = 194.925 Hz

Difference frequency is approx. 2 Hz

but bandwidth of analyser is 3.16 Hz ,

Hence the large errors at low frequencies.

The graph shows a typical curve of inharmonicity v (mode number)². The error bars represent 3·16 Hz of the B+K 2010 Analyser.

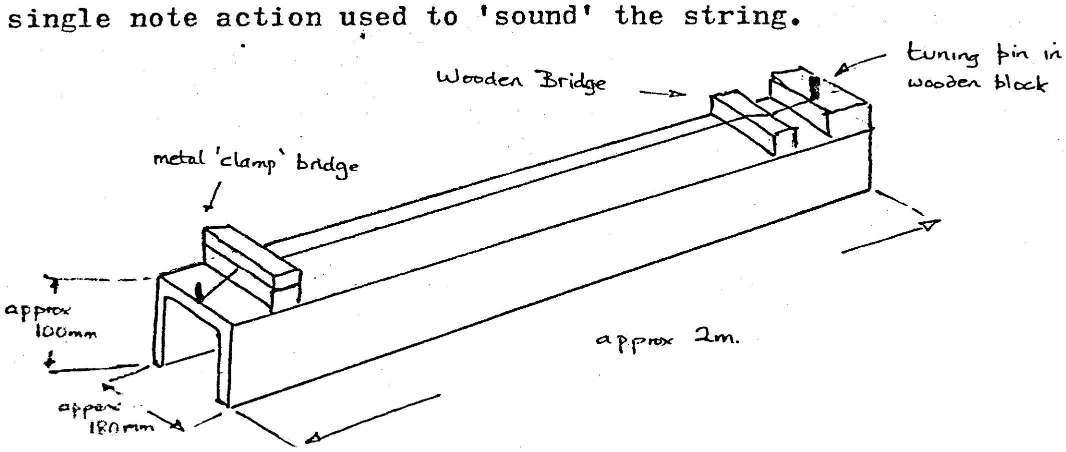


<u>Experiment to determine if the soundboard contributed significantly to string inharmonicity</u>

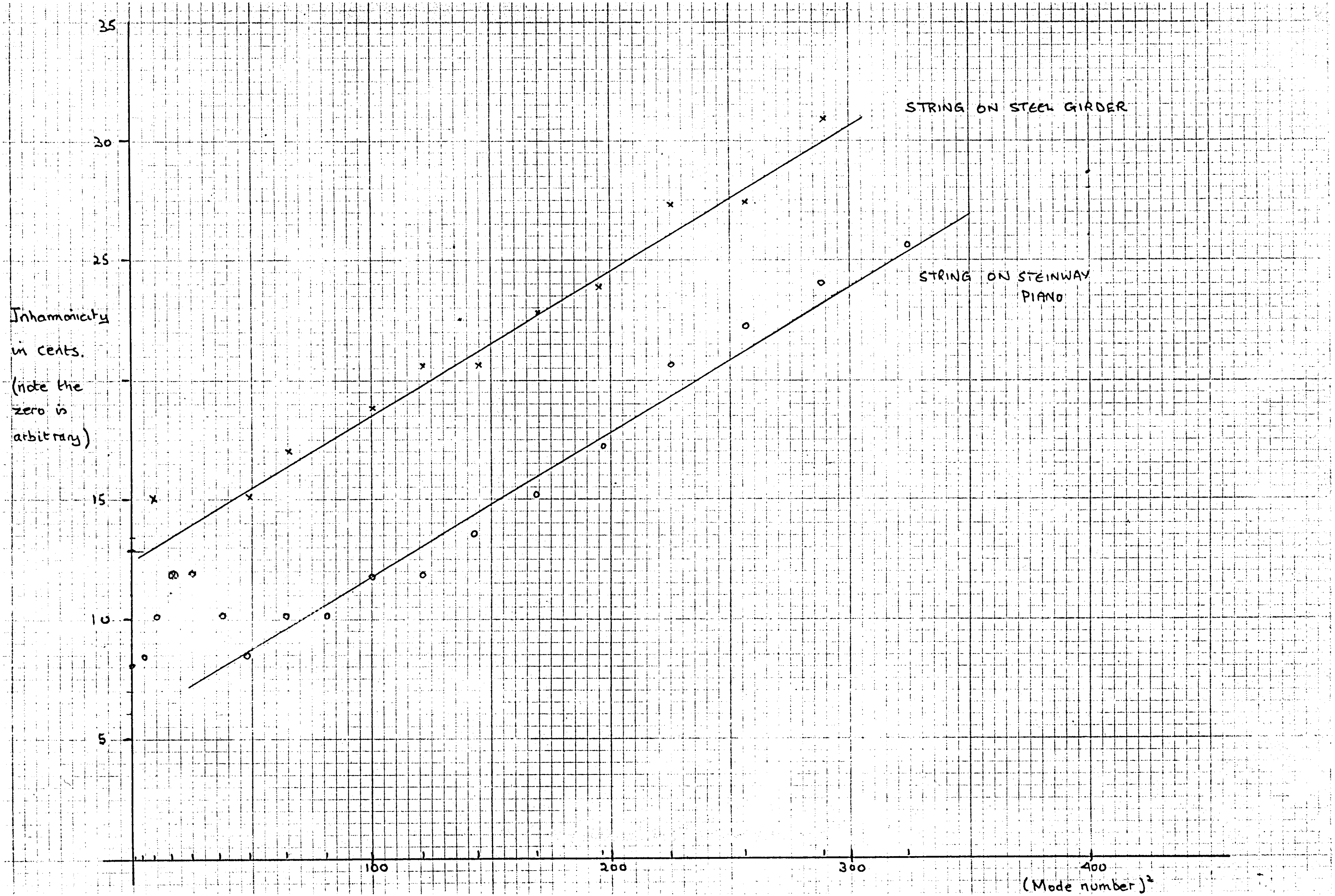
This experiment was carried out during the original work on inharmonicity, and was not included in the thesis. It was however discussed at the viva-voce examination, and it was advised that a brief description should be placed in the addenda.

The aim of the experiment was to compare the inharmonicity of a string on a soundboard with that of a similar string supported by a steel girder. The inharmonicity of the A2 plain wire string on a Steinway grand piano was measured, using the same techniques described in section 2.7.2 of the thesis except that an electromagnetic pick up was used in place of a microphone. The pick up was placed near the end of the string so as many partials could be recorded as possible.

A string of similar dimensions was then strung on an iron girder fitted with a metal bridge similar to that of the piano, and also a wooden bridge; the distance between the two being the same as on the grand piano. The same pick up as was used in the experiment on the piano was placed in a similar position and a single note action used to 'sound' the string.



The results obtained are shown graphically on page 0.20.
The results were also analysed using the calculator programme in appendix C 1 .


STRING ON STEEL GIRDER
STRING ON STEINWAY PIANO
Inhamonicity in cents. (note the zero is arbitrary)
(Mode number)²
35
30
25
15
10
5
100
200
300
400

From the calculator programme

Inharmonicity of string mounted on girder $B = 0.059$ cents n^{-2}

correlation coefficient 0.97

Inharmonicity of string on piano $B = 0.051$ cents n^{-2}

correlation coefficient 0.95

From these results it can be seen that there is no significant change in inharmonicity due to the soundboard. It should however be noted that the soundboard resonances will contribute 'random' changes to the frequencies of the partials depending upon the placing of the soundboard resonances. (see note on page 0.10 and fig 1.4.) A partial frequency which is slightly lower than a soundboard resonance will 'see' the soundboard as a 'springy' termination and hence the partial frequency of the string will be lowered.

A note on piano tuning practice

The system of piano tuning described in the thesis is the most common employed in the U.K., being that taught in colleges offering courses in piano technology. Some tuners, however, use a different system relying on higher harmonics. When tuning fifths instead of listening to a third partial and a near coincident second partial they listen to the sixth partial of the lower note and the fourth partial of the upper note. This leads to a slightly different scale, and if the practice of listening to higher harmonics is also used when tuning octaves a greater 'stretch' in tuning will occur compared to that achieved when single octaves are used.

A note on pianos used for duets

A simple experiment was carried out to confirm that differently scaled instruments lead to a different tuning. Two upright pianos, one an upright 'mini' piano, the other an older traditional style piano were tuned by the same experienced piano tuner. Whilst one piano was being tuned, the other had its strings damped to prevent any resonances from giving 'clues' to the tuner. The pianos were then compared note for note, slight differences in tuning being noted in the tuning octave. Outside the tuning octave the differences were much greater, and would be noticeable in performance.

When two identical 'mini' pianos were subjected to a similar test there was a much better agreement of the tuning, although the tunings were not identical. Hence the recommendation on page 5.3 concerning pianos for the performance of duets.

1 BASIC THEORY

1.1 Introduction

It is believed that Brook Taylor was the first to develop a 'correct' formulae relating the frequency, length, tension and mass for the vibrating string, in a paper published in 1713. In 1747, d'Alembert published a paper in which he derived the differential equation for wave propagation along a vibrating string. These papers have been published recently in a collection of early papers edited by LINDSAY (1973).
Most standard texts on acoustics contain a development of the wave equation, as the techniques required are between those applied to the simple oscillator (single mass and spring system) and those applied to the two dimensioned oscillator (such as plates and membranes). The most commonly cited texts are those of RAYLEIGH (1894) and HELMHOLTZ (1885).

If a string is stretched between two fixed points and the string is struck (or plucked) at some point along its length, the string will be seen to vibrate. This vibration is the result of transverse wave motions travelling along the string in opposite directions and successive reflections of the wave from the fixed ends of the string. The wave motion down the string may be simply observed if a rope is fixed at one end and the other end held in the hand. A disturbance sent down the rope will be seen to travel with a velocity dependant on the tension in the rope and to be reflected and inverted at the fixed end. If a transverse wave could be sent down the rope to a free end it would still be reflected, but not inverted. These reflected waves give rise to what is known as a standing wave on the vibrating string and to nodes and antinodes described in school physics textbooks and musical acoustics texts such as BACKUS (1970).

An alternative method of analysing the vibration of a string is to consider the string as being made up from a number of point masses equally spaced along the length of the string. This leads to the conclusion that the number of possible modes of transverse vibration is equal to the number of point masses. When the analysis is extended to an infinite number of masses an infinite number of transverse modes are found to be possible. This method of analysis is given in a diagrammatic form by BENADE (1976). This method is also suited to simulation of string vibration by computer,and was extended to take into account string stiffness by BACON & BOWSHER (197). An analysis by this method, although more complex, yields the same result as that by considering the propagation of a transverse wave on a string.

1.2 The wave equation for strings

The wave equation for strings was first derived by EULER (1748). The wave equation can be derived by considering the forces acting to restore a segment of the vibrating string to the equilibrium position.

The string is assumed to have

negligible stiffness

uniform linear density (μ)

rigidly supported ends

constant tension (T)

no loss of energy, either due internally to bending or externally to radiation.

These assumptions are generally accepted as being reasonable, provided the amplitude of vibration of the string is small and the string is thin, i.e. its length is very large compared with its diameter. The analysis does not take into account other forms of vibration such as longitudinal or torsional modes.

If the segment of the vibrating string is at a position x from an arbitrary datum point, and the segment has length δx, this can be represented as shown in fig. 1.1 . The tension in the string is T, and the transverse displacement is in the direction y.

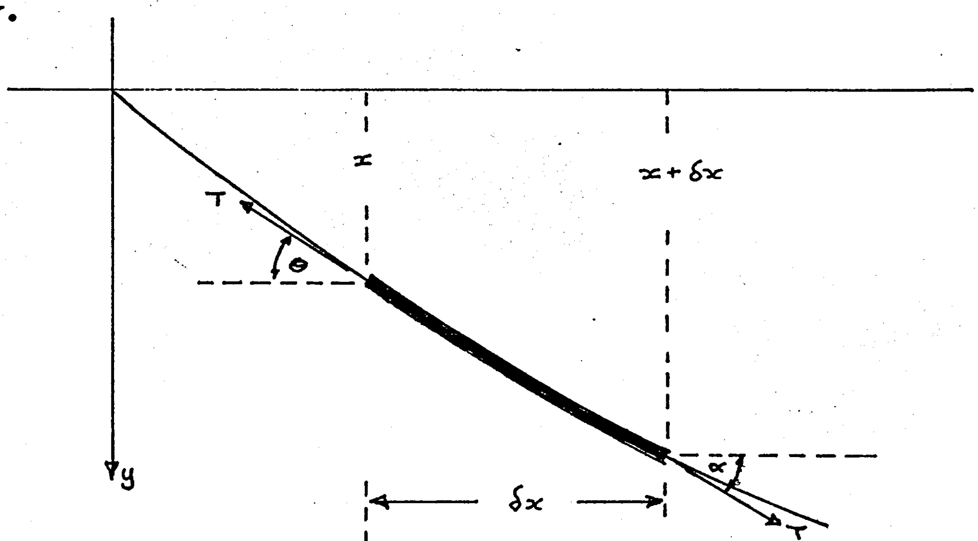


Fig 1.1

Resolving the forces in the y direction

Sum of the forces = (mass of segment) x (acceleration of segment)

If μ is the linear density, i.e. mass per unit length

$$T(\sin\alpha - \sin\theta) = \mu\,\delta x\ \partial^2 y/\partial t^2$$

If the angles α and θ are small , using the approximation $\sin x = x$, yields

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2} \qquad (1.1)$$

This shows that the transverse acceleration at a point on a string is directly proportional to the rate of change of the slope at that point.

Equation 1.1 is linear, since it involves only the first power of the derivatives which means that the principle of superposition may be applied, i.e. the total displacement of two or more component displacements is additive. Dimensional analysis shows that $\sqrt{\frac{T}{\mu}}$ has the dimension of velocity. The equation is known as the wave equation and represents a wave disturbance propagating along a string with a velocity $\sqrt{\frac{T}{\mu}}$, this will be shown later in this chapter.

1.3 Solutions of the wave equation

Equation 1.1 is a second order partial differential equation and hence its complete general solution consists of two arbitrary functions. Two such functions may be written

$$y = f_1\,(x-ct) \qquad \text{and} \qquad y = f_2\,(x+ct)$$

These may be combined, since the principle of superposition applies to give

$$y = f_1\,(x-ct) + f_2\,(x+ct) \tag{1.2}$$

Partial differentiation gives

$$\partial^2 y/\partial t^2 = c^2 f_1''\,(x-ct) + c^2 f_2''\,(x+ct)$$

and

$$\partial^2 y/\partial x^2 = f_1''\,(x-ct) + f_2''\,(x+ct)$$

where f'' is d^2f/dx^2

This gives the wave equation in the form

$$\partial^2 y/\partial t^2 = c^2\,\partial^2 y/\partial x^2 \tag{1.3}$$

The general solution for the wave equation (Eqn. 1.2) represents the propagation of an arbitrary waveform in the direction +x and also the direction -x. This is shown both mathematically and in diagrammatic form by HUNTER (1957). The positive x propagation is represented by the term

$$f_1\,(x - ct)$$

and the negative x propagation by the term

$$f_2\,(x + ct)$$

The velocity of propagation in both directions is c .
Comparing equations 1.1 and 1.3 gives

$$c = \sqrt{\frac{T}{\mu}} \tag{1.4}$$

It is possible to continue with the general solution, and impose boundary conditions, but it is more useful to consider a harmonic solution.
Such a solution may be written

$$y = A_1\, e^{j(\omega t - kx)} + A_2\, e^{j(\omega t + kx)} \tag{1.5}$$

where

$$k = \omega/\lambda \quad , \quad \omega = 2\pi f$$

λ is the wavelength and **k** the wavelength constant or wave number.

An alternative method of showing the velocity of propagation of a pulse on a string is given by MORSE (1948). If a string under constant tension T is subjected to deformation by a frictionless tube, the force exerted by the string on the tube is given by

$$F = T\theta - \mu\,\theta\,c^2$$

where θ is the angle subtended by an elemental arc of the deformation and c is the velocity of propagation of the deformation

The net force is thus zero when

$$c^2 = T/\mu$$

which agrees with eqn. 1.4

Hence the deformation set up by the tube will continue to exist without the presence of the tube as long as the velocity c persists. FEATHER (1961) quotes the above derivation as being first used in connection with the vibrating string by P.G. Tait in 1883.

1.4 The finite string

When a string is terminated at a fixed point equation 1.5 applies just as it does for the infinite string. However the complex constants A_1 and A_2 are no longer arbitrary since the displacement of the string at the termination must at all times be zero. This occurs if the two parts of the equation are equal and opposite. If the termination is at the point x=0, then equation 1.5 reduces to

$$\begin{aligned} y &= A\left\{e^{j(\omega t - kx)} - e^{j(\omega t + kx)}\right\} \quad (x>0) \\ y &= 0 \quad (x<0) \end{aligned} \qquad (1.6)$$

where A is complex.

Using Euler's relationship this may be written

$$y = 2A\,(\sin kx)\,e^{j\omega t} \qquad (1.7)$$

This no longer represents a wave travelling along the string, but a standing wave. That is each point of the string moves with simple harmonic motion, that points amplitude depending on the value of sin(kx) at that point.

If the other end of the string is terminated at x=L, then the displacement at this point must again be zero. In order that eqn. 1.7 should also satisfy this boundary condition

$$\sin kL = 0 \qquad \text{or} \qquad kL = n$$

where n is any integer.

Hence

$$\lambda = \frac{2L}{n}$$

This confines the harmonic vibration of the string to those modes where the half wavelength multiplied by some integer is equal to the length of the string. Therefore only certain frequencies of vibration are possible, these frequencies being given by

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}} \qquad (1{\cdot}8)$$

where n is the mode number

f the frequency of vibration

T the tension

μ the linear density, and L the length of the string.

This is often known as Mersenne's Law, since it confirms the laws of vibrating strings discovered by Mersenne in 1638. These are summarised by WOOD (1937) as follows ;

1. For a given string and a given tension the frequency varies inversely as the length (this principle was known to Aristotle).

2. For a string of given length and material the frequency varies as the square root of the tension applied.

3. The frequency of vibration of strings of the same length and subjected to the same tension, varies inversely as the square root of the mass μ per unit length of the string.

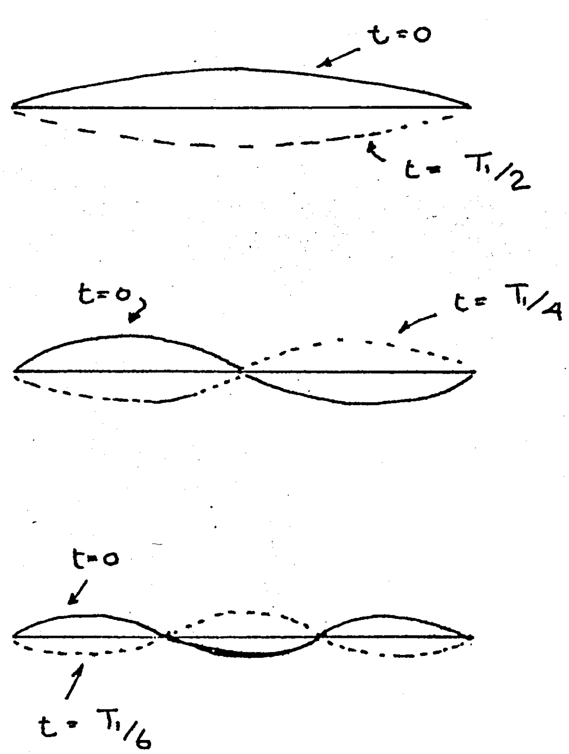
A discussion on practical versions of Mersenne's law is contained in Appendix 1 .

The general form for the instantaneous displacement for the n^{th} mode is given by

$$y_n = \left(A_n \cos \omega_n t + B_n \sin \omega_n t \right) \sin \frac{n\pi x}{L} \qquad (1.9)$$

The shape of the vibrating string is often represented as below to show displacement nodes and anti-nodes, where a node is a position of no displacement and an anti-node a position of maximum displacement.

T_1 is the period of the fundamental mode.



Fundamental mode

Second mode

Third mode

Fig 1.2

Since superposition applies, the displacement is given by the sum of any number of modes;

$$y = \sum_{n=1}^{n=\infty} \left\{ A_n \cos \omega_n t + B_n \sin \omega_n t \right\} \sin \frac{n\pi x}{L} \qquad (1.10)$$

Thus, the infinite string which is a medium for wave transmission becomes a medium for standing waves when confined by boundaries. However these standing waves may also be thought of as two waves travelling in opposite directions.

The solution (1.10) is quoted by RAYLEIGH (1894) as being originally derived by Daniel Bernoulli in 1755.

1.5.1 Characteristic impedance of the infinite string

The characteristic impedance of a string is defined as the ratio of transverse tension to the transverse velocity at any point on an infinite string carrying a progressive wave.

At each point on the string there is exerted an equal and opposite tension, if we take the tension exerted in the direction of travel of the wave and the wave is travelling in the positive x direction, then the tension is $-T\,\partial y/\partial x$

From the above definition

$$\text{Characteristic impedance} = Z_s = \frac{-T\,(\partial y/\partial x)}{\partial y/\partial t} \qquad (1.11)$$

Since the wave is travelling in the positive x direction; we find

$$Z_s = \sqrt{\mu T} = \frac{Tk}{\omega} = \mu c = r\sqrt{T\pi\rho} \qquad (1.12)$$

The characteristic impedance of a string is a real quantity, having no quadrature component, and is therefore purely resistive. It is proportional to the radius (or diameter) of the string and to the root of the tension, and independant of any driving force applied to the string, hence a characteristic of the string.

1.5.2 Input impedance of the finite string

For a string of finite length, the input impedance is found to be a function of the length of the string and the nature of the termination of the string. If the string is terminated at a fixed point then the wave equation solution

$$y = A\left\{e^{j(\omega t - kx)} - e^{j(\omega t + kx)}\right\} \tag{1.6}$$

holds true.

The input impedance (sometimes called the driving point impedance) is defined as the ratio of transverse tension to transverse velocity.

$$Z_i = \frac{-T\ \partial y/\partial x}{\partial y/\partial t} \tag{1.13}$$

This expression is the same as that for characteristic impedance eqn. 1.11 . If we now substitute from eqn. 1.6 we get

$$Z_i = \frac{kT}{\omega}\ \frac{e^{-jkx} + e^{jkx}}{e^{-jkx} - e^{jkx}}$$

This gives

$$Z_i = -j\frac{k}{\omega}T\ \cot kx \tag{1.14}$$

and substituting from eqn. 1.12

$$Z_i = \frac{j}{Z_s}\tan kx$$

The input impedance for a finite string terminated by a rigid support can be seen to be a pure reactance. Because Z_i is purely reactive the power input to the string must be zero - for a discussion of power see section 1.6 .

The above result is given by HUNTER (1957) . A similar result is given by MORSE (1948), but he expresses the result as an admittance, and has i=-j

$$Y_i = {}^{-i}/_{\mu c}\ \tan kx$$

$$= {}^{j}/_{Z_0}\ \tan kx \tag{1.15}$$

where $Z_0 = Z_s$

Below (fig 1.3) are shown graphs of velocity, tension and input impedance of a rigidly supported string as a function of string length. These graphs are taken from HUNTER(1957).

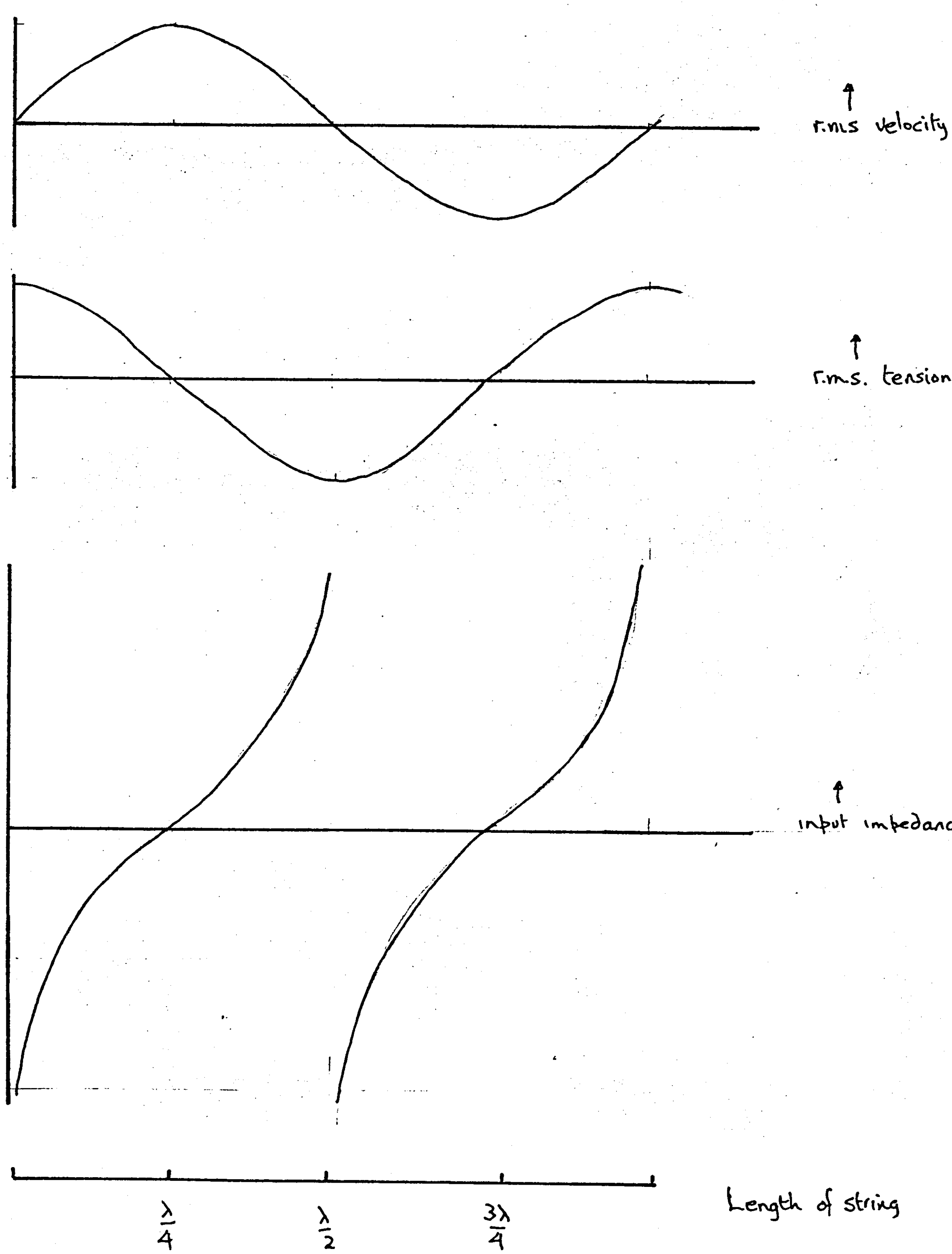


Fig 1.3

1.6 Energy of a vibrating string

The vibrating string may be thought of as a collection of elementary oscillators. The energy of each elementary oscillator of length δx is found by summing the kinetic and potential energies. If the displacement of the elementary length is given by

$$y = A_1 \sin \omega t$$

then the kinetic energy is given by

$$\tfrac{1}{2} m v^2 = \frac{\mu \delta x}{2} \omega^2 A_1^2 \cos^2 \omega t$$

The maximum value of kinetic energy being $\frac{\omega^2 A_1^2}{2} \mu \delta x$

When the kinetic energy has maximum value, the string is in the position y=0, and when y has a maximum value the string is at the position where y is a maximum, its velocity is zero and hence the kinetic energy is zero and potential energy a maximum. Since there is no loss mechanism, the maximum potential energy must have the same value as the maximum potential energy. The total energy of the vibrating element is therefore

$$\frac{\omega^2 A_1^2}{2} \mu \, dx$$

Due to its position along the string

$$A_1 = A \sin \frac{n\pi x}{L}$$

so the energy of the element may be expressed

$$dE = \tfrac{1}{2} \omega^2 A^2 \left(\sin^2 \frac{n\pi x}{L} \right) \mu \, dx$$

The energy assoc iated with a string of length L is

n x energy associated with length L/n

$$E = n \frac{\omega^2 A^2}{2} \mu \int_0^{L/n} \sin^2 \frac{n\pi x}{L} \, dx$$

$$E = \frac{\omega^2 A^2}{4} . M \qquad (1.16)$$

where M is the total mass of the string.

The space average energy density is analagous to the familiar root mean square (r.m.s.) average of amplitude in time. The space average energy density is an average over the length of the string, and since this repeats for each half wavelength along the string

$$\text{Space average energy density} = \mathcal{E} = \frac{1}{\lambda/2}\int_0^{\lambda/2} \frac{1}{2} A^2 \omega^2 \sin^2 \frac{\pi x}{L} \mu \, dx$$

$$= \frac{\mu A^2 \omega^2}{4} = \frac{E}{L} \qquad (1.17)$$

For a string, supporting a progressive wave the displacement maxima for the progressive wave all have equal value, A, changing phase from segment to segment having no effect on the energy. So eqn (1.17) becomes

$$\mathcal{E} = \frac{\mu \omega^2 A^2}{2} \qquad (1.18)$$

Since the amplitude of the standing wave is twice that of the components.

$$\text{Energy density} = \mathcal{E} = \mu \omega^2 A_s^2 \qquad (1.19)$$

where A_s is the amplitude of the standing wave.

The rate of energy flow per unit length or power flow for a wave travelling at a velocity c, is given by

$$\mathcal{E}_c = \tfrac{1}{2} \mu c V^2$$

$$= \tfrac{1}{2} \sqrt{T\mu}\, V^2$$

$$= \tfrac{1}{2} Z_s V^2 \qquad (1.20)$$

where V is the velocity amplitude in the y direction of the particle on the string. The power flow is a real quantity as Z_s is purely resistive.

The above is based on a derivation given by HUNTER (1957), in which there is an error - the line above his eqn.2.72 should read

$$E = \frac{\mu \omega^2 A^2 L}{2\pi} \left\{ \frac{n\pi x}{2L} - \frac{1}{4} \sin \frac{2n\pi x}{L} \right\}_0^{L/n}$$

A similar result for the total energy of a vibrating string, eqn 1.16 above, is given by MORSE (1948), but he uses a harmonic solution and does not give the simplified case above. Morse has

$$\text{Total energy} = \sum_{n=1}^{\infty} 2\pi^2 \left(\frac{L\mu}{2}\right) f_n^2 A_n^2$$
$$= \sum_{n=1}^{\infty} \frac{M \omega_n^2 A_n^2}{4}$$

which agrees with eqn 1.16 above.

Here the energy is expressed as a series of terms, each term depending on one of the harmonic components. The energy of the string is equal to the total energy of an infinite number of harmonic oscillators, each having a massequal to half the mass of the string with a frequency f_n and an amplitude A_n.

If Mersenne's Law, eqn 1.8 is substituted into eqn. 1.21

$$E = \sum_{n=1}^{\infty} \frac{\pi^2 T A_n^2}{4L} \cdot n^2$$

which agrees with the result given by FEATHER (1961).

Hence the total energy is proportional to the string tension and independant of the mass of the string.

Also, the energy of each 'harmonic' $\propto (A_n n)^2$

Thus to give a weighted harmonic diagram or power spectrum (see BACKUS (1970) p203) the amplitude of the harmonics must be multiplied by the mode number n.

1.7 Differences between real and ideal strings

In the preceding derivations effects of string stiffness have not been taken into account. A brief examination of the strings in the top octave of a piano will show that the description of short stiff bars may be more appropriate. A typical top string of a piano has a length of 55 mm and a diameter of 0.8 mm. The effect of string stiffness is discussed in section 2.1 and the sections following.

The string vibration has been assumed to have been in a single plane, no account has been taken of the other possibilities. Rotational oscillation is possible about the rest position of the string, also torsional vibration about the axis of the string. Oscillation is also possible in the longitudinal mode. These possibilities could be the subject of further investigation.

It has also been assumed that all strings have a uniform linear density. Wound strings are generally non-uniform as the winding is often terminated at some point along the vibrating length of the string, only the core forming the strings from that point to the point where the strings crosses the bridge. A single value for density for plain strings of the same material has also been assumed, but this if inadequate. FENNER (1959) states that the thinner strings, due to more frequent drawing in the manufacturing process, have a higher average density than the thicker strings, the difference being in the order of 4%.

Piano bass strings are not light when their mass is compared to their tension. Effects of mass should also be the subject of further investigation. This condition is also violated in the case of harpsichords, although the mass of the strings is less than that of piano strings, the tension in harpsichords is in the range 21 - 88 Newtons, some 4 - 10 % of the tension found in pianos.

The supports for the strings in keyboard instruments are not strictly rigid. The wrest plank bridge may be nearly rigid, but the soundboard bridge must move in order to transmit vibration to the soundboard for radiation. A simplified way of looking at this problem is shown below in figure 1.4. The effects of bridge motion can be seen to cause either a lengthening of the string or a shortening of the string, either being the case for each mode of a single string depending on the characteristics of the soundboard at that frequency.

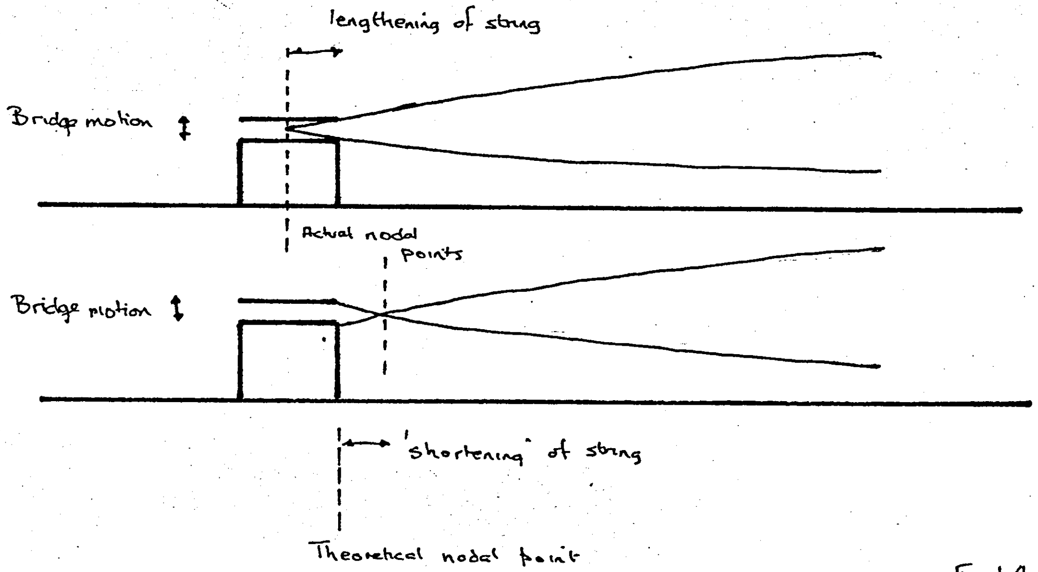


Fig 1.4

It has also been assumed that the string parameters are not affected by the application of tension - both the static tension and the dynamic tension must have some effect on the string. An investigation into the effects of Hookes Law with reference to string length, and Poissons Ratio with reference to changes in string diameter may be useful. Initial investigations have shown that changes in diameter are of measureable amplitude.

From the above simple consideration of the vibrating string the amplitude nodes are points of rest. In the diagram below are shown the displacement nodes and anti-nodes for the first three nodes of vibration.

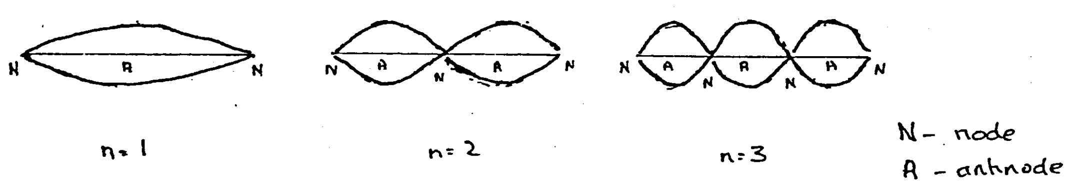


Fig 1.5

It has been pointed out by WOOD(1937) that the nodes are not points of rest as small motions must exist as the vibrational energy is required for the maintenance of vibration beyond these points. He quotes results obtained by S.Ray in 1926 who found that the velocity of transverse waves along a string is not an independant constant but a function of $\frac{A}{\lambda}$ where A is the amplitude of the oscillation.

Wood also quotes experiments by Raman in 1909 which showed that the motion at the anti-nodes tends to be 'figure of eight' and the motion at the assumed nodal points tend to be rather flat parabolas.

It is evident that for some purposes a simple treatment of the motion of a vibrating string is insufficient and a more thorough analysis is required.

2 INHARMONICITY

2.1 Introduction

It is well known that the 'harmonic' components of the motion of the vibrating piano string are not truly harmonic. The 'harmonic' component frequencies are not integer multiples of the fundamental frequency of vibration, this is known as inharmonicity.

In the following discussion, the frequency components of the vibrating string are referred to as partials, the term harmonic being reserved for the special case of partials conforming to a harmonic series.

Inharmonicity is caused mainly by the lack of flexibility of the piano string. This lack of flexibility or stiffness causes a progressive raising in frequency of the partials from the harmonic series. The frequencies of the partials of a stiff string may be predicted from equations derived by DONKIN (1884), and developed by others, e.g. RAYLEIGH (1894) and MORSE (1948). The theory developed by these authors is restricted to the case of plain wire strings.

The earliest experimental determination of inharmonicity in piano strings was by SHANKLAND and COLTMAN (1939). Further experiments were carried out by SHUCK and YOUNG (1943), and later by FLETCHER (1964). These experiments were not only confined to the plain wire strings, but extended to wound piano strings.

2.2 Bending of a stiff string

As the ratio of string diameter to string length is increased, the stiffness becomes more important in determining the behaviour of the string.

For a string vibrating with small amplitude, the curvature of the string, defined as $1/R$ may be represented by d^2y/dx^2. The increase in length of the curved string may be assumed negligible. The bending moment M is the couple required to bend the string so that the string forms the arc of a circle radius R.

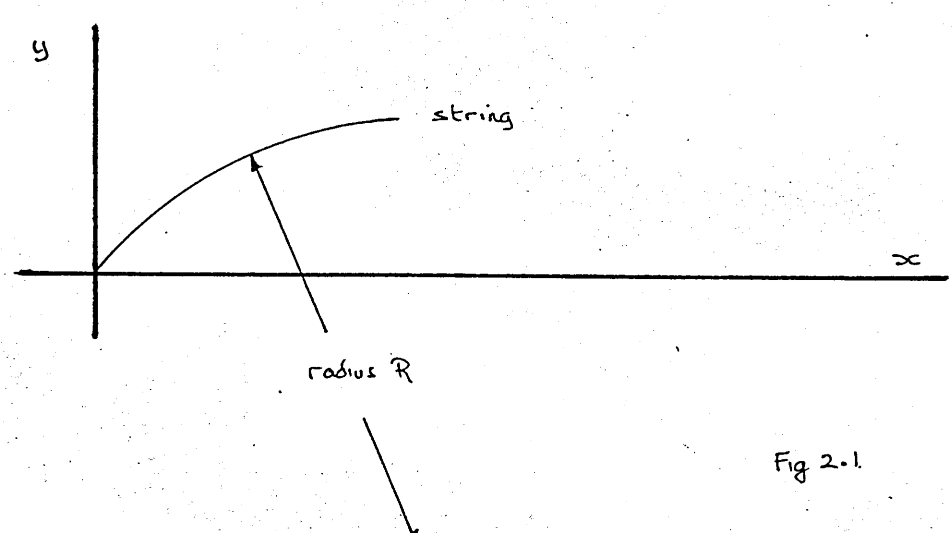


Fig 2.1.

The strain in any part of the string is proportional to the curvature.

By Hookes law

$$M \qquad 1/R$$

Referring to Fig. 2.2, the neutral plane is neither extended or contracted. Considering a filament ab at a distance r from the neutral plane with cross sectional area w and of length δx

After bending the length increases to $ab' = \delta x + \Delta$

The force stretching the filament $= \dfrac{Ew\Delta}{\delta x}$

$$= \frac{Ewr}{R}$$

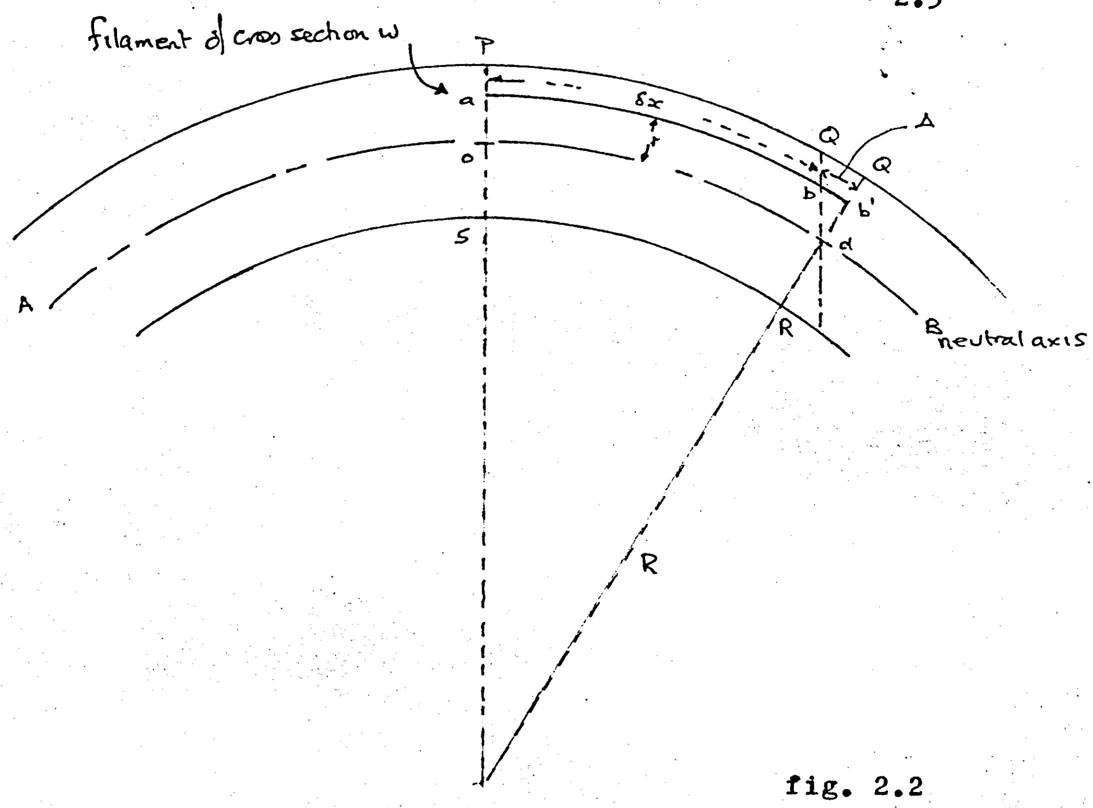


fig. 2.2

The moment of this force about the neutral axis is $\dfrac{E w r^2}{R}$

The total moment M of all horizontal and tensile compressive stresses perpendicular to PS of area S about cd

$$M = \frac{E}{R} \sum w r^2 = \frac{E.S\kappa^2}{R}$$

where $S\kappa^2$ is the second moment of area.

Hence $M = ES\kappa^2 \, d^2y/dx^2$

If the rotational acceleration of the segment is assumed to be zero, we have

$$\frac{dF}{dx} = ES\kappa^2 \, \partial^4 y/\partial x^4$$

Since the forces F and $F + \dfrac{\partial F}{\partial x}\partial x$ are in opposite directions

the net force $F = -ES\kappa^2 \, \partial^4 y/\partial x^4$. Eqn. 2.1.

2.3 Modification of the wave equation

The wave equation derived in section 1.2 allows only for a force restoring the displaced string to its rest position due to the tension. From equation 1.1 this restoring force is $T\,\partial^2 y/\partial x^2$.

FLETCHER,H (1964) states that an additional restoring force of the form $R\,\partial y/\partial t$ could be introduced. This is due to frictional forces, but he states that the term would produce only a very small effect on the frequencies of the partials of a modern piano string. The frictional restoring force would affect mainly the decay of vibration of the string. This is further discussed in a paper by FLETCHER,N.H.(1976).

The force due to elastic stiffness however is significant. The restoring force due to stiffness is derived in section 2.1, and is given as

$$F = -E\,S\,\kappa^2\,\partial^4 y/\partial x^4 \qquad \text{Eqn. 2.1}$$

FLETCHER,H (1964) states that this term was first given in the above form by RAYLEIGH (1894).

The wave equation (eqn. 1.1) can be modified by including the term for stiffness, giving

$$-ES\kappa^2\,\partial^4 y/\partial x^4 + T\,\partial^2 y/\partial x^2 = \mu\,\partial^2 y/\partial t^2 \qquad \text{Eqn. 2.2}$$

where E is Youngs modulus of elasticity

S is the cross sectional area of the string $= \pi d^2/4$

T is the tension

u is the linear density

κ is the radius of gyration $= d/4$

and d is the diameter of the string

2.4 Solution of the modified wave equation

The solution of the wave equation may be assumed to have the form

$$y = A e^{2\pi(kx - jft)} \qquad (2.3)$$

If substitution is made into equation 2.2, and the following substitutions also made

$$f_o = \frac{1}{2L}\sqrt{\frac{T}{\mu}} \qquad (2.4)$$

and

$$\beta^2 = \frac{\pi E S k^2}{T L^2} = \frac{\pi^3 d^4 E}{64 L^2 T} \qquad (2.5)$$

the modified wave equation becomes

$$k^4 - \frac{k^2}{4\beta^2 L^2} - \frac{f^2}{16\beta^2 L^4 f_o^2} \qquad (2.6)$$

Hence there are four possible values for k, given by

$$k = \pm k_1 , \qquad K_1^2 = \frac{1}{8\beta^2 L^2}\left\{\left(1 + \frac{4\beta^2 f^2}{f_o^2}\right)^{\frac{1}{2}} + 1\right\} \qquad (2.7)$$

$$k = \pm j k_2 , \qquad K_2^2 = \frac{1}{8\beta^2 L^2}\left\{\left(1 + \frac{4\beta^2 f^2}{f_o^2}\right)^{\frac{1}{2}} - 1\right\} \qquad (2.8)$$

where k_1 and k_2 are related by the expression

$$k_1^2 - k_2^2 = 1/4\beta^2 L^2 \qquad (2.9)$$

The general solution, eqn 2.3, above becomes

$$y = e^{-2\pi j f t}\left\{K_1 \cosh 2\pi k_1 x + K_3 \cos 2\pi k_2 x + K_2 \sinh 2\pi k_1 x + K_4 \sin 2\pi k_2 x\right\} \qquad (2.10)$$

This is eqn 6 given by FLETCHER,H (1964), where there is a misprint having a sin term following K_2 instead of sinh as above.

These relationships are independant of the boundary conditions.

For each value of k_1 there is a corresponding value of k_2 which may be found from eqn. 2.9 , and the corresponding frequency f from equation 2.6.

If the convention is adopted that the string has length L, with the origin of axes at the centre of the string, the string may be represented as in fig. 2.3.

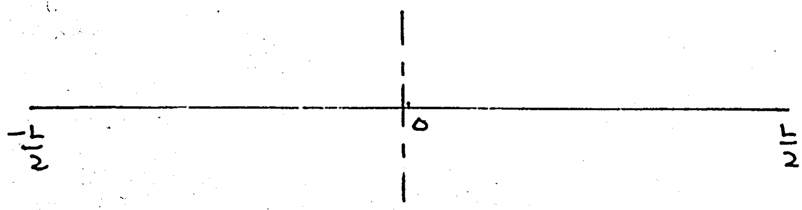


Fig 2.3

The solution of the modified wave equation will yield odd and even functions. The even functions will come from the first two terms of eqn. 2.10, and the odd functions from the last two terms.

2.5.1 The stiff string with pinned boundary

At a pinned boundary, the displacement of the string (at $x = \frac{L}{2}$ and $x = -\frac{L}{2}$) is zero. The string also has zero change of slope at these points, that is an elemental length considered at these points will be straight.

Taking the even functions only, $K_2 = K_4 = 0$, and the boundary conditions fit if K_1 is also zero and $\cos 2\pi k_2 \left(\frac{L}{2}\right) = 0$. This gives

$$k_2 = n/_{2L} \qquad \text{where } n = 1,3,5,7\ldots \quad \text{any odd integer} \tag{2.11}$$

From this the frequencies of the odd numbered partials is given by

$$f_n = n f_0 \left\{ n^2 \beta^2 + 1 \right\}^{1/2} \tag{2.12}$$

for these frequencies, the displacement is given by

$$y = e^{-2\pi j f t} \left\{ K_3 \cos 2\pi \frac{n}{2L} x \right\}$$

Taking the real part of the expression

$$y = K_3 \cos \left\{ \frac{\pi n x}{L} \right\} \cos 2\pi f_n t \qquad (2.13)$$

where K has a different value for each odd partial.

Taking the even functions, $K_1 = K_3 = 0$

and a similar result is obtained.

$k = n/_{2L}$ where n = 2,4,6..... any even integer

and again equation 2.12 results.

2.5.2 The stiff string with clamped boundary

At a clamped boundary the displacement of the string is zero. The string will also have zero slope at the boundaries. This leads to a solution for the allowable values of k_1 and k_2 which must be found using numerical methods. The frequencies of the partials is then given by

$$f_n = 2 k_2 f_0 \left\{ 1 + 4\beta^4 k_2^2 \right\}^{1/2} \qquad (2.14)$$

It is expected that the value of k_2 would be close to that obtained in the previous section and so k_2 may be written

$$k_2 = \frac{n}{2L(1+\epsilon}$$

where ϵ is a small value compared to unity.

Since from experimental work β has a small value the higher powers may be neglected, also the higher powers of ϵ .

This leads to the equation first developed by SEEBECK quoted by FLETCHER (1964).

$$f_n = n f_0 \left\{ 1 + (4/_\pi)\beta + \left[(12/_{\pi^2}) + n^2 \right] \beta^2 \right\}^{1/2} \qquad (2.15)$$

From this equation can be seen that the effect of clamping the ends of the string will raise slightly the frequencies of the partials. FLETCHER (1964)further modifies the equation to give

$$f_n = n f_0 \left\{ 1 + (2/\pi)\beta + \left[(4/\pi^2) + (n^2/2) \right] \beta^2 \right\} \qquad (2.16)$$

which is the form given by MORSE (1948).

In a piano, the string boundary condition lies between the pinned and clamped condition, and for either condition the equation

$$f_n = n F \left\{ 1 + \beta^2 n^2 \right\}^{1/2} \qquad (2.17)$$

may be used, where F and β are constants obtained from experiment.

If inharmonicity is defined as the difference in frequency between the ideal string and a string with stiffness (i.e. the departure of the partial frequencies from the harmonic series) then Inharmonicity = $f_n / n f_0$.

Inharmonicity is usually expressed in cents where 100 cents is equal to an equal tempered semitone, and 1200 cents an octave.

Inharmonicity = $\delta_n = 1200 \log_2 f_n / n f_0$

$$= \frac{1200}{\ln 2} \ln \frac{f_n}{n f_0}$$

Expanding equation 2.17 by the binomial theorem

$$\delta_n = \frac{1200}{\ln 2} \ln \left\{ 1 + \frac{\beta^2 n^2}{2} \right\}$$

$$\delta_n = B n^2 \qquad (2.18)$$

where B is the inharmonicity factor in cents. $= \frac{1200}{\ln 2} \frac{\beta^2}{2}$ (2.19)

YOUNG (1952) quotes the fractional error in δ_n resulting from the approximations as less than 1% if δ_n is less than 35 cents.

2.6 Review of other published work

The above equation (2.17) has been used in various forms by other authors. Generally the simplification of substituting f_o for F is made, where f_o is the fundamental frequency for the same string having no stiffness. Thus the equation being used is that for the stiff string with pinned boundary conditions, resulting in lower calculated values of frequency of the partials compared with those obtained using equation (2.17).

Expanding equation (2.12) using the binomial expansion

$$f_n = n \; f_o \left(1 + \frac{n^2 \beta^2}{2} \right) \qquad (2.20)$$

which may be written

$f_n = n \; f_o \; (1 + J n^2)$ which is the form given by BENADE (1976).

This equation is given in expanded form by WOOD (1955)

$$f = \frac{n}{2L} \sqrt{\frac{T}{\mu}} \left\{ 1 + \frac{n^2 \pi^3 E r^4}{8 T L^2} \right\} \qquad (2.21)$$

which is the contracted form due to RAYLEIGH (1894) as quoted by SHANKLAND & COLTMAN (1939) in a paper which also deals with the change of frequency of a string due to increased tension during the initial part of the vibration.

PARSONS (1970) gives an interesting method by which β^2 may be evaluated. He notes that the transverse frequency of vibration of a flexible string may be written

$$f_o = \frac{1}{2L} \sqrt{\frac{4T}{\pi \rho d^2}} \qquad (2.22)$$

Also the fundamental frequency of longitudinal vibration of a flexible string may be written

$$f_L = \frac{1}{2L} \sqrt{\frac{E}{\rho}} \qquad (2.23)$$

Substitution in equation (2.5) yields

$$\beta^2 = \frac{f_L}{f_0} \frac{\pi}{4} \frac{d}{L} \tag{2.24}$$

Hence β may be evaluated from the dimensions of the string and the frequencies of the fundamental transverse and longitudinal modes of vibration. The transverse mode may be excited by plucking the string and the longitudinal mode by stroking with a rosined cloth.

2.7 Experimental work on piano inharmonicity

The earliest experimental determination of inharmonicity was by SHANKLAND and COLTMAN (1939) using plain wire strings on a monochord. This was followed by SHUCK and YOUNG (1943) and later by FLETCHER (1964) who used a piano and extended their investigations to the wound strings of the piano.

2.7.1 Review of other experimenters work

The experiments by SHANKLAND and COLTMAN (1939) were carried out using a monochord and plain wire strings. Each partial was investigated separately, the string vibration being maintained by a feedback system. A signal picked up from the string at its supported end was amplified and passed back through the string as a large electrical current, the motor effect being achieved by placing a permanent magnet near the string. This method would work for strings of both magnetic and non-magnetic materials, provided the string was an electrical conductor.

Their results showed that the partial frequencies lay between those predicted for a clamped end condition and those for a pinned end condition, contrary to Rayleigh's assumption that the frequencies would be closer to the latter condition. They conclude that inharmonicity is a necessary part of musical instrument tone causing constantly changing phase relationships between the partials, and state that the observed inharmonicity is consistent with data on piano tuning obtained by RAILSBACK (1938) and the observations of D.C.Miller in an unpublished study on piano strings.

In their study SHUCK and YOUNG (1943) used a piano fitted with a vibration pick-up feeding a chromatic stroboscope to measure the frequencies of the partials. They present their results in a more practical form than SHANKLAND and COLTMAN (1939) expressing the inharmonicity of the partials in cents sharper than the expected harmonic. They also measured the decay of each individual partial and show photographs of three dimensional models of piano sound, i.e. amplitude, time, mode number. They note that the inharmonicity in cents is proportional to the square of the mode number (eqn 2.18) and conclude from their graphs that within limits this is approximately true. They extend their measurements to both plain and wound strings and show a graph of the variation of inharmonicity across the keyboard. From this infromation they go on to show the implications in piano tuning both within the temperament octave and across the full range of the piano. This is further discussed in the next chapter.

In a paper by FLETCHER (1964) the experiments carried out by SHUCK and YOUNG (1943) are repeated, but the graphs show more experimental data, a greater number of strings being measured. For example he shows the partial frequencies for the three strings of a trichord 'unison' , and includes more details on wound strings. He concludes that there is probably an optimum value for the inharmonicity for each string, but that value has not yet been found. The optimum value is not, in his opinion, zero, i.e. all strings harmonic.

2.7.2 Author's experimental work

The apparatus available was as follows-

An upright piano, manufactured by Barratt and Robinson, with a model 988 Challen frame and Kastner Wehlau action.

B&K 1" microphone type 4145

B&K heterodyne analyser type 2010

Dawe analyser type 1461A

Racal VLF counter with 0.1 Hz resolution

Nagra series 3 tape recorder, tape speed 381 mm/s

Texas Instruments programmable calculator type TI57.

The piano was tuned and regulated by an experienced piano technician. The microphone was placed 1 metre from the back of the soundboard and after amplification using the preamplifier section of the 2010 analyser the signal was recorded on the tape recorder at a tape speed of 381 mm/s. Since a mechanical striker for the piano keys was unavailable the notes were struck by hand using approximately equal force for each note. The piano tones were recorded with only one string sounding, the other strings being muted with felt wedges. (see fig. 2.3)

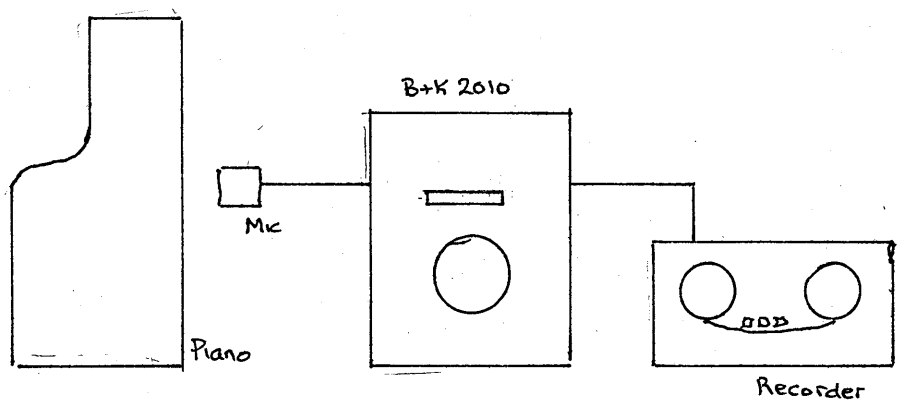


Fig. 2.3

Each note on the tape was then made into a loop of approximately 2 seconds duration and arranged to play repeatedly from the tape recorder.

Initially an attempt was made to measure the frequencies of the partials using a frequency counter. The signal from the tape recorder was fed into the Dawe analyser which acted as a filter the centre frequency of which was adjusted manually for maximum output and the resulting sinusoidal output fed into the VLF counter. This proved satisfactory at low frequencies, but unsuitable at higher frequencies due to the bandwidth of the filter. (see fig. 2.4)

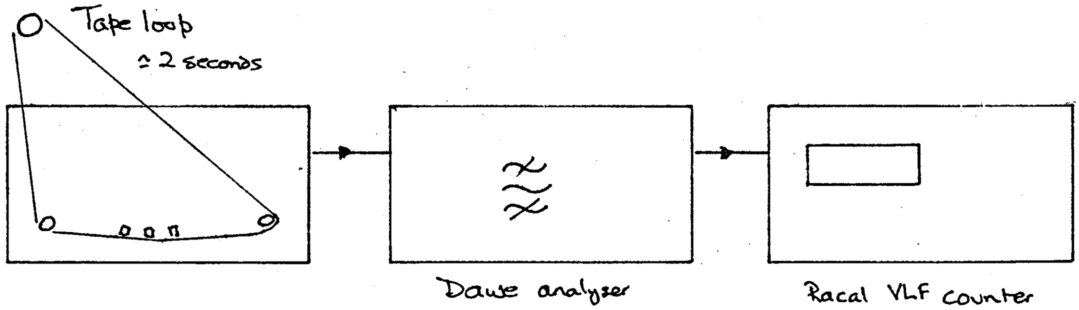


Fig 2.4

The bandwidth of the Dawe analyser is between 5% and 8% of the tuned frequency. Since the B&K 2010 analyser is of the heterodyne type its bandwidth is constant, and on this model may be adjusted down to 3.16 Hz. It was decided to use the B&K 2010, its disadvantage being that the manual tuning needs to be exact, whereas slight tuning errors of the Dawe filter did not affect the frequency reading.

The final choice of apparatus is shown in figure 2.5.

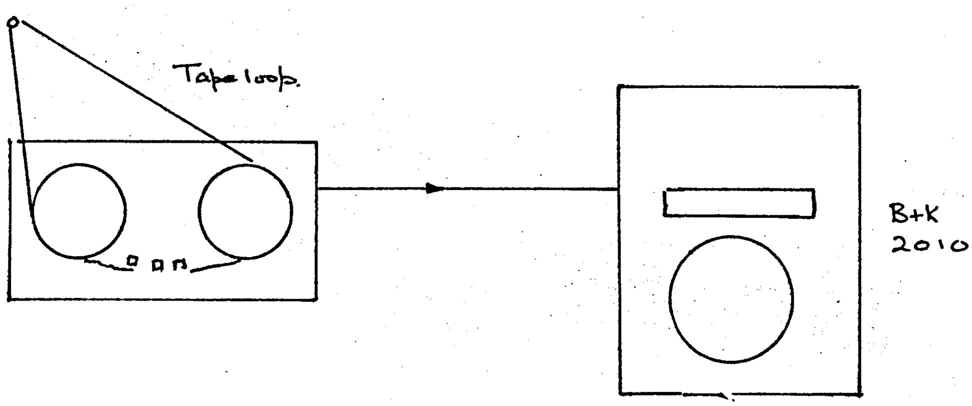


Fig. 2.5

Treatment of results

The frequencies of the partials for each string investigated was tabulated. Since eqn. 2.17 applies we can use

$$\delta_n = B \quad n^2$$

which is the same as eqn. 2.18, but now

$$\delta_n = 1200 \log_2 \frac{f_n}{nF} \qquad 2.25$$

Since F is unknown, a graph of n^2 against δ_n' could be plotted, where δ_n' is the deviation from the harmonic series of a nominal fundamental frequency f_{nom}, chosen to give positive values of inharmonicity.

A sketch of such a graph is shown in figure 2.6.

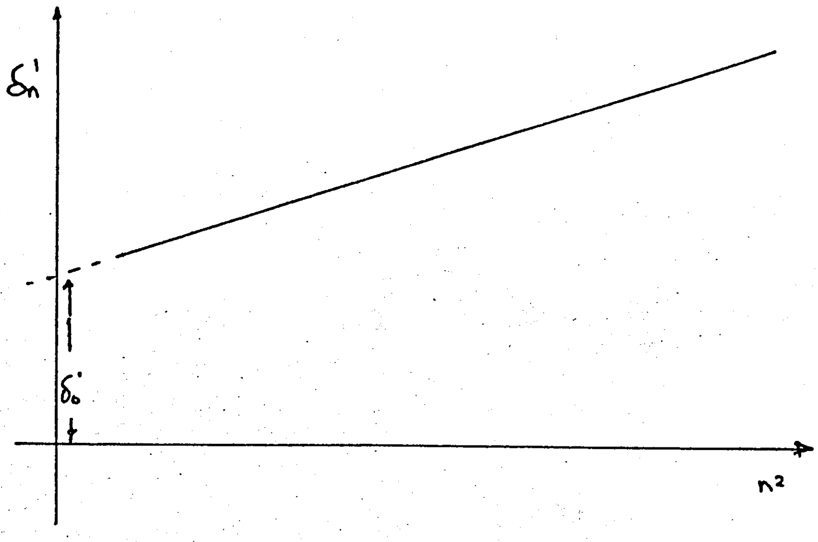


fig 2.6

F may then be calculated from

$$F = f_{nom} \times 2^{\delta_0'} \qquad 2\text{-}26$$

where δ_0' is the intercept on the axis.

It was decided rather than to plot graphs, a statistical programme using the TI 57 calculator be used. This would give more reliable results for the slope and intercept than hand plotting and give the additional information of the correlation coefficient. The programme is shown in the appendix of computer and calculator programmes beginning on page C1.

The results of the experiments are tabulated on the following pages. The frequencies are in Hz, and the inharmonicity is expressed in cents, the positive values indicating that the partial is sharper than the harmonic of the nominal fundamental.

Table of partial frequencies for an upright piano

C1 string, wound, length 1065 mm, nominal frequency 32.4 Hz

Mode number n	frequency f_n	inharmonicity δ'_n	n^2
1	33	31.8	1
2	65	5.3	4
3	98	14.2	9
4	131	18.6	16
5	162	0	25
6	196	14.2	36
7	230	24.3	49
8	262	18.6	64
9	295	20.1	81
10	329	26.5	100
11	365	41.3	121
12	399	44.8	144
13	435	55.8	169
14	470	61.5	196
15	507	73.2	225
16	545	86.6	256
17	581	92.4	289
18	619	103.1	324
19	658	115.3	361
20	696	123.7	400

Intercept 6.53 cents, hence F = 32.5 Hz

Slope = B = 0.293 cents n^{-2}

Correlation coefficient = 0.98

C2 string, wound, length 995 mm, nominal frequency 65 Hz

Mode number n	frequency f_n	inharmonicity δ_n'	n^2
1	65	0	1
2	130	0	4
3	197	17.7	9
4	261	6.6	16
5	326	5.3	25
6	390	0	36
7	455	0	49
8	521	3.3	64
9	686	3.0	81
10	652	5.3	100
11	720	12.1	121
12	786	13.3	144
13	854	18.3	169
14	923	24.6	196
15	992	29.9	225
16	1062	36.2	256
17	1132	41.8	289

Intercept -0.659 cents, hence F = 64.975 Hz

Slope = B = 0.128 cents n

Correlation coefficient = 0.89

C3 string, wound, length 925.5mm, nominal fundamental 129 Hz

Mode number	frequency	inharmonicity	n^2
n	f_n	δ_n'	
1	130.5	20	1
2	261	20	4
3	390	13.4	9
4	516	0	16
5	648	8	25
6	778	8.9	36
7	910	13.4	49
8	1039	11.7	64
9	1137	17.8	81
10	1302	16	100
11	1436	20.6	121
12	1570	24.4	144
13	1700	23.6	169
14	1839	31.3	196
15	1976	36.3	225
16	2119	45.5	256
17	partial too weak to measure		
18	2390	50	324

Intercept 8.23 cents, hence F = 129.6 Hz

Slope = B = 0.121 cents n

Correlation coefficient = 0.90

C4 string, plain, diameter 1.025mm, length 644mm, nominal freq. 259.5 Hz

Mode number n	frequency f_n	inharmonicity δ_n	n^2
1	259.5	0	1
2	520	3.3	4
3	779	1.1	9
4	1039	1.7	16
5	1300.5	4	25
6	1563	6.7	36
7	1828	10.9	49
8	2093	14.1	64
9	2365	21.7	81
10	2641	30.4	100
11	2910	33.3	121
12	3180	36.3	144
13	3464	45.8	169
14	3748	53.9	196
15	4036	62.7	225

Intercept -1.53 cents, hence F = 259.3 Hz

Slope = B = 0.282 cents n

Correlation coefficient 0.99

C5 string, plain, diameter 0.95mm, length 343.4 mm, nom. freq. 520 Hz

Mode number n	frequency f_n	inharmonicity δ_n'	n^2
1	520	0	1
2	1040	0	4
3	1565	5.5	9
4	2081	0.8	16
5	2628	18.5	25
6	partial too weak to measure		
7	3726	40.4	49
8	4316	62.1	64
9	4885	74.2	81

Intercept -5.59 cents, hence F = 518.3 Hz

Slope = B = 0.989 cents n

Correlation coefficient = 0.988

C6 string, plain, 0.9mm diam, length 183mm, nominal frequency 1040 Hz

Mode number n	frequency f_n	inharmonicity δ_n'	n^2
1	1040	0	1
2	2092	10	4
3	3164	24.2	9
4	4266	43.6	16

Intercept -2.17 cents, hence F = 1038.7 Hz

Slope = B = 2.883 cents n

Correlation coefficient 0.9994

Values of B for each string investigated were then plotted on a graph together with calculated values for the plain strings. The calculated values were calculated from equations 2.4, 2.5 and 2.19 which may be combined and rewritten

$$B = \frac{1200}{\ln 2} \frac{\pi^2}{128} \frac{E}{\rho} \frac{d^2}{L^4} \frac{1}{f_0^2} \qquad (2.27)$$

String	Measured B	Calculated B
4 C1	0.293	
16 C2	0.128	
28 C3	0.121	
34 F#3		0.286
40 C4	0.282	0.321
46 F#4		0.538
52 C5	0.989	0.852
64 C6	2.883	2.37
76 C7		6.56
88 C8		16.8

The resulting graph is shown on page 2.22, fig 2.7.
From the graph it may be seen that the calculated and measured values are in fair agreement.
The value of the inharmonicity factor may be seen to vary across the piano, values for wound strings decreasing to blend with the lower values of the lower plain strings. The author suggests that this is an important aspect of piano design, and that a smooth transition of inharmonicity values is important.

A further aspect of the graph fig 2,7 is discussed in section 3.1.

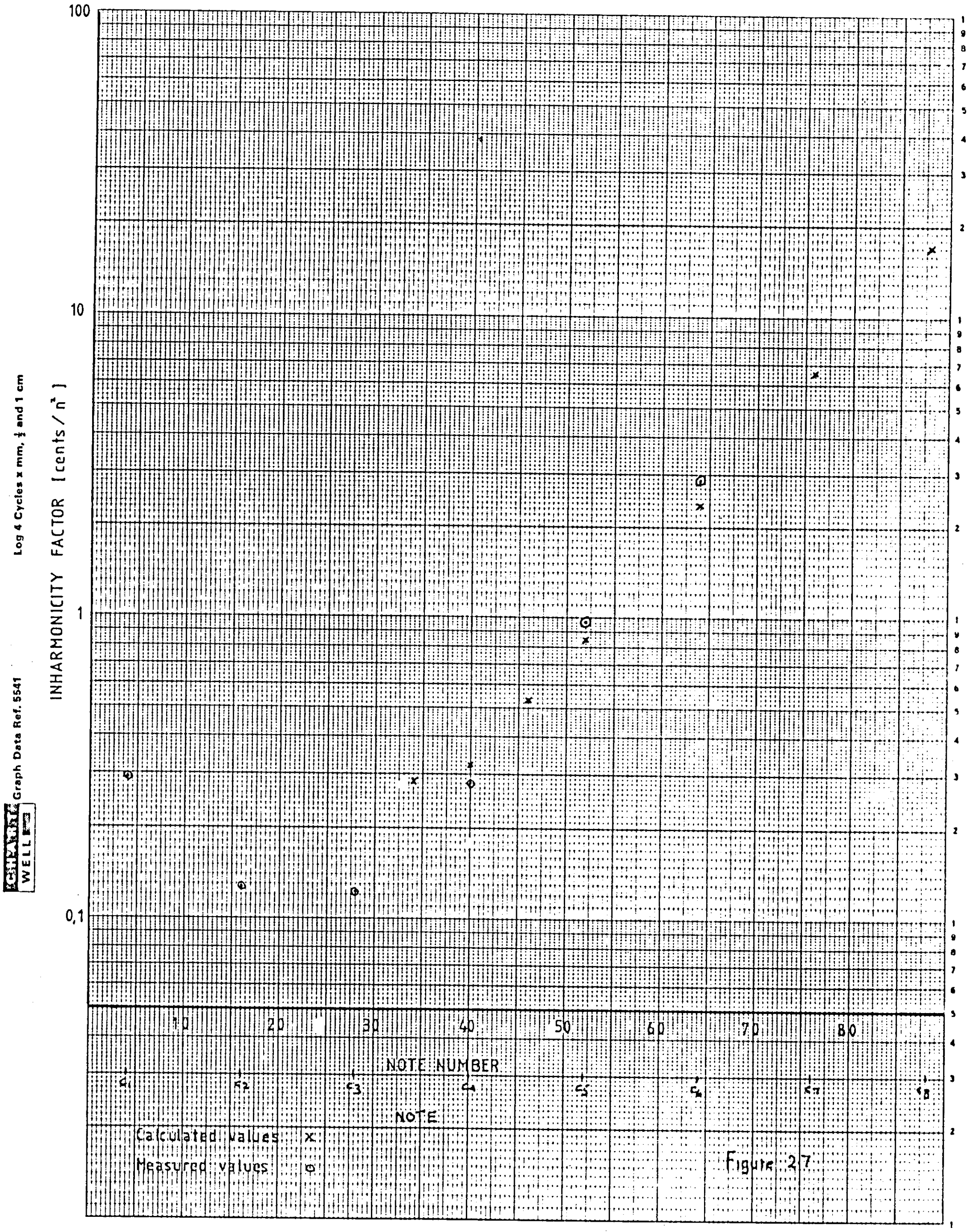

100
10
1
0,1
INHARMONICITY FACTOR [cents/n²]
10
20
30
40
50
60
70
80
NOTE NUMBER
C1
C2
C3
C4
C5
C6
C7
C8
NOTE
Calculated values x
Measured values ⊙
Graph Data Ref. 5541
Log 4 Cycles x mm, ½ and 1 cm

Figure 27

2.7.3 Comparison of author's work with other experimenters work

The earliest experimental determination of inharmonicity in piano strings was by SHANKLAND and COLTMAN (1939). They experimented with electrically maintained vibrations of brass and steel strings supported by a laboratory jig and did rot extend their experiments to a string on a musical instrument.

SCHUCK and YOUNG (1943) used a piano in their experiments which were carried out on F strings, both plain and wound. Their conclusions were mainly concerned with the effects of inharmonicity oh tuning, which the author discusses in section 3.

YOUNG (1952) again used a piano in his experiments, and as in his previous experiments used a chromatic stroboscope to measure frequency.Unlike his previous experiments he concentrated on the plain strings only, but did use three pianos two grand and one uproght all by the same manufacturer. He notes that for the upright piano a supporting bar in the piano frame caused a discontinuity in the graph of inharmonicity. He states that O H Schuck believed that the evolution of piano design caused the smooth change of inharmonicity from string to string and that good piano tone should result from a piano with low inharmonicity throughout, which is borne out by the preference for the grand piano he examined which had low inharmonicity and smooth inharmonicity changes. It is the author's opinion that the discontinuity shows a weakness in the design of the upright piano examined by R W Young.

FLETCHER (1963) outlines three methods of measuring inharmonicity, and recomends electrical excitation with the use of a frequency counter for low frequencies and a chromatic stroboscope at higher frequencies. His experiments included measurements on both wound and plain strings on an upright piano. He extended the work on plain strings to include measurements of the three strings of a tri-chord. It is the authors opinion that these three strings should have similar inharmonicity factors achieved by having the strings of equal length and diameter. This would mean ensuring that small sections of the bridges should be as near parallel as possible.

In all cases, for the upright piano, the earlier experimenters showed similar results to those obtained by the author.

3 IMPLICATION OF INHARMONICITY TO TUNING THEORY AND PRACTICE

3.1 Introduction

The modern piano is according to most textbooks such as WHITE (1972) tuned to equal temperament. The usual procedure is to tune single strings in a middle octave, then to tune the remaining notes in the tri-chords. The tuning is then extended to the upper and lower notes by tuning octaves from the middle octave, occasionally using wider intervals as checks.

The middle octave chosen is usually that which extends from F3 to F4. Referring to the graph fig. 2.7 it will be noted that this choice is such that tuning is carried out on those strings which have a minimum of inharmonicity.

To tune the middle octave, or 'lay the bearings' as it is usually called , the C4 string is tuned to a tuning fork. Since the fork generally used has a frequency corresponding to C5 this means that the second partial of the C4 string is tuned beatless or in unison with the fork. The F3 string a fifth below C4 is the next string tc be tuned. In order to produce an equal tempered scale this fifth has to be narrowed. This is on order to make a cycle of fifths come to exactly a whole number of octaves. Twelve pure fifths just exceed seven octaves

$$1.5^{12} > 2^{7}$$

Another way of considering the problem is to consider that an octave being twelve semitones may be represented as an interval of 1200 cents, a fifth being seven semitones 700 cents. A pure fifth is represented by the ratio 3:2, and since

$$\text{cents in an interval} = 1200 \log_2(\text{frequency ratio})$$

a pure fifth is represented by 701.955 cents. Hence the equal tempered interval is narrower than the pure or beatless interval. The fifth is narrowed by allowing the second partial of the higher string to beat with the third partial of the lower string. The calculation of the beat rate necessary is discussed in section 3.2.

Laying the bearings continues with the tuning of G3 a tempered fourth below C4. This time the interval has to be widened as an interval of a fourth and fifth must add up to a pure octave.

In cents: octave = fourth + fifth

therefore, fourth = octave - fifth

fourth = 1200 - 701.955 = 498.045 cents.

The remainder of the sequence for laying the bearings is given in appendix A5, on page A9. This appendix also covers some suggestions on the use of electonic aids for the teaching of piano tuning.

The use of electronic aids for tuning pianos is a debatable area. From the rest of the discussion in this section it will be seen that such aids must take inharmonicity into account. Few commercial devices do this, but BEDFORD (1972) discusses a tuning aid which allows for the inharmonicity of the piano being tuned, and which will vary the compensation in each individual case.

3.2 Computer calculation of beats for an ideal string

Beats are formed from the difference frequency between near coincident partials. In the case of an ideal string the partials form a true harmonic series and hence can be referred to as harmonics. The beat rates may be calculated by constructing a table of the notes of the equal tempered scale and subtracting the frequencies of the appropriate harmonics.

A computer programme was written in Basic for a Research Machines 380Z computer and stored on a floppy disc. A listing for the programme may be found in appendix C2, page C3. The programme was run, and a print out of the results are on pages 3.4 and 3.5.

The programme uses a code for the notes in the octave F3 to F4, where

F = 0

F# = 1

G = 2

G# = 3 etc.

After the programme has been loaded, the computer asks for the required frequency of A4, this is at line 150. The computer then calculates the frequency of A3, and lines 210 to 300 produce a table of fundamental frequencies from F3 to F4 and their harmonics up to the eigth.

In order to find the near coincident harmonics from the table the computer uses the appropriate information contained in lines 680 to 740. Lines 380 to 540 select the number of semitones in the interval and the beat rates are calculated from lines 740 to 840.

```
READY:
RUN

EQUAL TEMPERED SCALE AND HARMONICS
FOR F3 TO F4
INPUT A4 FREQUENCY AND TYPE RETURN

435 Hz NORMAL CONTINENTAL ,QUEENS HALL
439    NEW PHILHARMONIC
440    BRITISH STANDARD
444    MEDIUM,OLD RSA
452    KNELLER HALL
454    OLD PHILHARMONIC
? 440

A4 HAS FREQUENCY 440 Hz
```

174.61	349.23	523.84	698.46	873.07	1047.68	1222.30	1396.91
185.00	369.99	554.99	739.99	924.99	1109.98	1294.98	1479.98
196.00	392.00	587.99	783.99	979.99	1175.99	1371.98	1567.98
207.65	415.30	622.96	830.61	1038.26	1245.91	1453.57	1661.22
220.00	440.00	660.00	880.00	1100.00	1320.00	1540.00	1760.00
233.08	466.16	699.25	932.33	1165.41	1398.49	1631.57	1864.66
246.94	493.88	740.82	987.77	1234.71	1481.65	1728.59	1975.53
261.63	523.25	784.88	1046.50	1308.13	1569.75	1831.38	2093.00
277.18	554.37	831.55	1108.73	1385.91	1663.10	1940.28	2217.46
293.66	587.33	880.99	1174.66	1468.32	1761.99	2055.65	2349.32
311.13	622.25	933.38	1244.51	1555.63	1866.76	2177.89	2489.02
329.63	659.26	988.88	1318.51	1648.14	1977.77	2307.39	2637.02
349.23	698.46	1047.68	1396.91	1746.14	2095.37	2444.60	2793.83

fig 3.1

```
TYPE RETURN TO GIVE BEAT RATES
FOR MINOR THIRDS
?
F  TO G# MINOR THIRD -9.423  BEATS NARROW
F# TO A  MINOR THIRD -9.983  BEATS NARROW
G  TO A# MINOR THIRD -10.58  BEATS NARROW
G# TO B  MINOR THIRD -11.21  BEATS NARROW
A  TO C  MINOR THIRD -11.87  BEATS NARROW
A# TO C# MINOR THIRD -12.58  BEATS NARROW
B  TO D  MINOR THIRD -13.33  BEATS NARROW
C  TO D# MINOR THIRD -14.12  BEATS NARROW
C# TO E  MINOR THIRD -14.96  BEATS NARROW
D  TO F  MINOR THIRD -15.85  BEATS NARROW
```

```
TYPE RETURN TO GIVE BEAT RATES
FOR THIRDS
?
F  TO A  THIRD  6.929  BEATS WIDE
F# TO A# THIRD  7.341  BEATS WIDE
G  TO B  THIRD  7.778  BEATS WIDE
G# TO C  THIRD  8.241  BEATS WIDE
A  TO C# THIRD  8.731  BEATS WIDE
A# TO D  THIRD  9.25  BEATS WIDE
B  TO D# THIRD  9.8  BEATS WIDE
C  TO E  THIRD  10.38  BEATS WIDE
C# TO F  THIRD  11  BEATS WIDE


TYPE RETURN TO GIVE BEAT RATES
FOR FOURTHS
?
F  TO A# FOURTH  .7892  BEATS WIDE
F# TO B  FOURTH  .8361  BEATS WIDE
G  TO C  FOURTH  .8858  BEATS WIDE
G# TO C# FOURTH  .9385  BEATS WIDE
A  TO D  FOURTH  .9943  BEATS WIDE
A# TO D# FOURTH  1.053  BEATS WIDE
B  TO E  FOURTH  1.116  BEATS WIDE
C  TO F  FOURTH  1.182  BEATS WIDE


TYPE RETURN TO GIVE BEAT RATES
FOR FIFTHS
?
F  TO C  FIFTH -.5912  BEATS NARROW
F# TO C# FIFTH -.6264  BEATS NARROW
G  TO D  FIFTH -.6636  BEATS NARROW
G# TO D# FIFTH -.7031  BEATS NARROW
A  TO E  FIFTH -.7449  BEATS NARROW
A# TO F  FIFTH -.7892  BEATS NARROW


TYPE RETURN TO GIVE BEAT RATES
FOR MINOR SIXTHS
?
F  TO C# MINOR SIXTH -11  BEATS NARROW
F# TO D  MINOR SIXTH -11.65  BEATS NARROW
G  TO D# MINOR SIXTH -12.35  BEATS NARROW
G# TO E  MINOR SIXTH -13.08  BEATS NARROW
A  TO F  MINOR SIXTH -13.86  BEATS NARROW


TYPE RETURN TO GIVE BEAT RATES
FOR SIXTHS
?
F  TO D  SIXTH  7.924  BEATS WIDE
F# TO D# SIXTH  8.395  BEATS WIDE
G  TO E  SIXTH  8.894  BEATS WIDE
G# TO F  SIXTH  9.423  BEATS WIDE

*BREAK @ LINE 900
```

In order to print out note names in the beat rate tables a subroutine in lines 1010 to 1140 is used.

0 is converted to	70 32 which is the ASCII code for	F
1	70 35	F#
2	71 32	G
3	71 35	G#

etc.

From the tables of results it will be seen that for equal temperament the following intervals are tuned narrow,

minor thirds

fifths

minor sixths

and the following intervals tuned wide

thirds

fourths

sixths.

Since the octaves are perfect, it may be noticed that an octave can be made from a wide tuned interval plus a narrow tuned interval; e.g. a fourth plus a fifth

a third plus a minor sixth

a sixth plus a minor third.

It should also be noted that the beat rate frequency increases by the constant factor $\sqrt[12]{2}$ as each interval is raised by a semitone, i.e. the beat rate increases smoothly and doubles when an interval is raised an octave.

3.3 Calculation of beat rates for stiff strings

(fundamentals tuned to equal temperament)

From the discussion in section 2, it is apparent that for stiff strings the partials are no longer harmonic. Hence re-calculation of the beat rates is neccessary if the piano fundamentals are to remain tuned to equal temperament.

Since further access to the computer was not possible a programme was written for the TI 57 programmable calculator.

The stiffness factor β^2 was calculated using

$$\beta^2 = 4.15 \times 10^6 \frac{d^2}{L^4} \frac{1}{f_1^2} \qquad \text{eqn. 3.1}$$

which is derived from equations 2.4 and 2.5. The following values were used,

$E = 210 \times 10 \ N\,m^{-2}$

$\rho = 7800 \ kg\,m^{-3}$

Values of β are tabulated in fig 2.3, page 3.9.

The calculator programme to determine the partial frequencies of the stiff string is listed in appendix C3, page C5. The programme is based on equation 2.17.
Using this programme the table of partial frequencies fig. 3.3, page 3.9 was constructed.

From the table of partial frequencies the beat rates for the various intervals were calculated, by taking the difference in frequency between the relevant partials. A negative result indicating a narrow interval and a positive result a wide interval. The results are tabulated on pages 3.10, 3.11 and 3.12.

From the table of results it will be seen that the beat rates no longer form a uniform pattern. The beat rates are dependant on the values of o which in turn is dependant on the string lengths and diameter, so the beat rates are a function of the design of the piano being considered.

Although the octaves are still pure as far as the fundamental frequencies are concerned, they will now be beating intervals. This is due to the higher partials being non-harmonic. F3 to F4 is now narrow by 0.2 beats for the 2nd partial of F3

1.1	3rd
3.7	6th
8.5	8th

Similarly for other intervals, the higher order beat rates form a non-harmonic progression, whereas for the ideal string the progression is harmonic.

<u>Table of string stiffness and equal tempered frequencies</u>

Note	β^2	f_1
F 3	3.3295726×10^{-4}	174.614 Hz
F♯3	3.3087129×10^{-4}	184.997
G 3	2.7014713×10^{-4}	195.998
G♯3	2.6697524×10^{-4}	207.652
A 3	2.9170532×10^{-4}	220.000
A♯3	3.0708219×10^{-4}	233.082
B 3	3.3836728×10^{-4}	246.942
C 4	3.7033265×10^{-4}	261.626
C♯4	4.1121501×10^{-4}	277.183
D 4	4.3717415×10^{-4}	293.665
D♯4	4.8239430×10^{-4}	311.127
E 4	5.3461224×10^{-4}	329.628
F 4	5.7836828×10^{-4}	349.228

Figure 3·2

<u>Partial frequencies of stiff string tuned to equal temperament</u>

NOTE	Fundamental f_1	Partials 2nd	3rd	4th	5th	6th	7th	8th
F 3	174.61	349.40	524.54	700.20	876.55	1053.77	1232.02	1411.48
F♯3	185.00	370.18	555.72	741.82	928.65	1116.39	1305.22	1495.32
G 3	196.00	392.15	588.63	785.58	983.16	1181.53	1380.85	1581.27
G♯3	207.65	415.47	623.62	832.27	1041.58	1251.72	1462.85	1675.12
A 3	220.00	440.19	660.77	881.92	1103.84	1326.72	1550.74	1776.09
A♯3	233.08	466.38	700.10	934.47	1169.70	1405.99	1643.55	1882.60
B 3	246.94	494.13	741.83	990.27	1239.71	1490.40	1742.57	1996.47
C 4	262.63	523.54	786.04	1049.41	1313.93	1579.8[illegible]	1847.58	2117.27
C♯4	277.18	554.71	832.92	1112.14	1392.73	1675.02	1959.33	2245.99
D 4	293.66	587.71	882.53	1178.50	1476.00	1775.41	2077.1[illegible]	2381.44
D♯4	311.13	622.70	935.18	1249.00	1564.61	1882.45	2202.93	2526.54
E 4	329.63	659.78	991.00	1323.79	1658.67	1996.18	2336.80	2681.04
F 4	349.23	699.06	1050.10	1402.95	1758.21	2116.46	2478.28	2844.24

Figure 3.3

Stiff strings tuned to equal temperament

Beat rates for minor thirds.

(6th partial of lower note beating with 5th partial of upper note)

F to	G♯	- 12.2 beats narrow
F♯	A	- 12.6
G	A♯	- 11.8
G♯	B	- 12.0
A	C	- 12.8
A♯	C♯	- 13.3
B	D	- 14.4
C	D♯	- 15.3
C♯	E	- 16.4
D	F	- 17.2

Beat rates for thirds

(5th partial of lower note beating with 4th partial of upper note)

F to	A	+ 5.4 beats wide
F♯	A♯	5.8
G	B	7.1
G♯	C	7.8
A	C♯	8.3
A♯	D	8.8
B	D♯	9.3
C	E	9.9
C♯	F	10.2

Beat rates for fourths

(4th partial of lower note beating with 3rd partial of upper note)

F to	A♯	- 0.1 beats narrow
F♯	B	+ 0.01 beats wide; almost a pure interval
G	C	+ 0.5 beats wide
G♯	C♯	0.7
A	D	0.6
A♯	D♯	0.7
B	E	0.7
C	F	0.7

Beat rates for fifths

(3rd partial of lower note beating with 2nd partial of upper note)

F to	C	- 1.0 beats narrow
F♯	C♯	- 1.0
G	D	- 0.9
G♯	D♯	- 0.9
A	E	- 1.0
A♯	F	- 1.0

Beat rates for minor sixths

(8th partial of lower note beating with 5th partial of upper note)

F	to C♯	- 18.8 beats narrow
F♯	D	- 19.3
G	D♯	- 16.7
G♯	E	- 16.5
A	F	- 17.9

Beat rates for sixths

(5th partial of lower note beating with 3rd partial of upper note)

F	to	D	+ 6.0 beats wide
F♯		D♯	6.5
G		E	7.8
G♯		F	8.5

3.4 Prediction of piano tuning temperament using traditional beat rates

Of the three programmes presented by the author, this is the most important. Pianos are tuned using the beat rates calculated in section 3.2 to a very close approximation. One of the checks used by a tuner is to ensure that there is an even increase in beat rate as an interval moves up the keyboard, and that over one octave the beat rate does double.

A computer programme was written in Basic for a Research Machines 380Z computer and stored on floppy disc. A listing for the programme may be found in appendix C3, page C7. The programme was run, and a print out of the results is on page 3.15.

The programme uses a similar code for notes as the previous computer programme, see page 3.3. except F=1, F#=2 etc.

The harmonic beat rates are calculated and stored in array C for fifths and Q for fourths. The number following the array designation indicates the lower note of the interval, hence

C1 contains the beat rate for the fifth F to C

C2 F♯ C♯ etc. and

Q1 fourth F to A♯

Q2 F♯ B etc.

These calculations are performed in lines 340 to 600.

Lines 650 to 710 hold the β^2 values for the piano used. For other pianos lines 780 to 870 may be used, requiring an input of string length and diameter to calculate β^2 using equation 3.1 which is contained in line 2020.

The frequency of A3 is written into the programme at line 920, and the partials of this string are calculated in lines 920 to 960.

Lines 970 to 1230 contain the tuning sequence, and the subroutine at line 2050 is used when tuning a fifth up, and the subroutine at line 2150 used when tuning a fourth down. lines 1240 to 1310 contain the print instructions for tabulating the frequencies of the fundamentals and partial frequencies. The calculation of cents deviation from equal temperament is performed by lines 1315 to 1370.

From the results on page 3.15 agraph of deviation of piano temperament from equal temperament was drawn. This is shown on page 3.16 . The slope of a straight line drawn on this graph was found to be 2.2 cents per octave with a correlation coefficient of 0.265 using the calculator programme in appendix C1. SCHUCK and YOUNG (1943) show a similar graph, which from inspection shows better correlation and a slope of 1.5 cents per octave. This reflects their use of a medium grand piano which has longer string lengths than the author's upright piano, and hence has lower β^2 values.

SCHUCK and YOUNG (1943) suggest that this stretch of tuning in the middle octave indicates that the stretched octaves in the upper and lower registers of the piano is a natural result of string inharmonicity. They reproduce a graph of stretched tuning after Railsbeck.

```
RUN
HARMONIC BEAT RATES
C 1           -.59121654742
C 2           -.62637211196
C 3           -.6636181362
C 4           -.70307892375
C 5           -.74488617201
C 6           -.78917940892

Q 1            .78917941172
Q 2            .83610646241
Q 3            .88582394086
Q 4            .93849777337
Q 5            .99430375453
Q 6            1.053428133
Q 7            1.1160682291
Q 8            1.1824330986
```

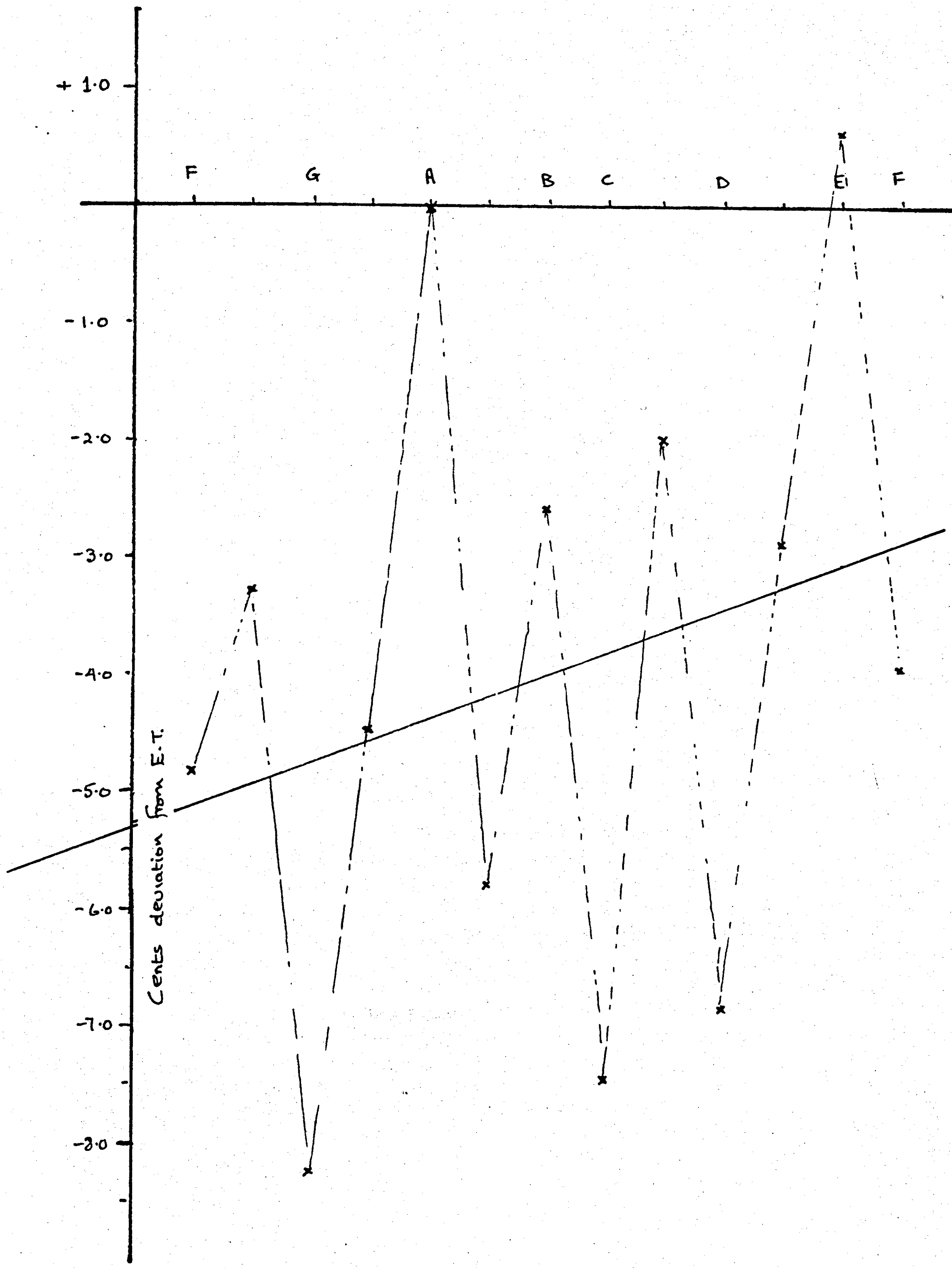
f_o , and partials of stiff strings of piano

```
  174.10   174.13   348.43   523.08   698.25
  184.62   184.65   369.48   554.69   740.46
  195.03   195.07   390.35   586.07   782.43
  207.07   207.12   414.49   622.37   831.02
  219.97   220.00   440.19   660.77   881.92
  232.25   232.30   464.94   698.25   932.57
  246.51   246.57   493.54   741.30   990.24
  260.43   260.50   521.46   783.32  1046.52
  276.81   276.86   554.07   831.95  1110.86
  292.45   292.51   585.40   879.06  1173.87
  310.53   310.61   621.67   933.62  1246.92
  329.66   329.75   660.02   991.36  1324.27
  348.33   348.43   697.46  1047.70  1399.75

PIANO TEMPERAMENT DEVIATION FROM EQUAL TEMPERAMENT
 1            -4.8241363296
 2            -3.2673920105
 3            -8.2432383521
 4            -4.4739800262
 5             1.0913936421E-09
 6            -5.790183908
 7            -2.5800832011
 8            -7.4395455025
 9            -1.995992971
 10           -6.8239155236
 11           -2.8852251904
 12            .63292182822
 13           -3.9602149602

*BREAK @ LINE 2010
READY:
```

Deviation from equal temperament

Slope ≈ 2·2 ¢ per octave

correlation coefficient 0·265



4 Dynamic behaviour of piano tones

4.1 Introduction

In addition to the fact that the piano's tones are generally inharmonic, the partials of any particular note tend to vary considerably in loudness. In the case of the piano this variation is called the partial structure. For a sound composed of true harmonics the term harmonic structure is more generally used.

The partial structure of piano tone has been investigated by FLETCHER, BLACKHAM & STRATTON (1962), initially using a sonograph and later using a conventional narrow band analyser. They show graphs for the first six partials showing the relative amplitudes for the first few seconds after the string has been struck.

Simple spectra, probably averaged over a short period of time, are shown by OLSON (1967). These show that the lower piano notes are richer in partials than the higher notes, but do not give much information.

The theoretical basis for the spectra of strings was given by RAYLEIGH(1894) and is repeated in slightly differing forms by most standard textbooks. BENADE (1976) gives a summary of the main points and differences between plucked and struck strings in his discussion on the piano, clavichord and harpsichord.

Experimental determination of the decay times for the complete piano tone was first investigated by MARTIN (1947). The typical decay for piano tones, except the top two octaves, was found to be a double decay. These decays are expressed in terms of a time constant, being the time that would be required for a decay of 60dB. MARTIN (1947) reported that the decay rates of adjacent notes may vary by as much as 3:1. His paper is mainly concerned with differences between an upright piano, a baby grand piano and an electronic piano - the electronic piano being similar to a conventional instrument but with the soundboard replaced by pick-ups and an amplifier.

A survey of the literature yielded no information on harpsichord decay times.

FLETCHER (1976) summarises the main causes for decay from a theoretical standpoint, giving the main causes as the effects of end termination, and for the string itself air and internal damping.

4.2.1 Initiation of vibration by plucking the string

If the initial displacement of a string is produced by plucking, that is pulling the string to one side by a narrow plectrum and suddenly releasing the string, the waveform which travels along the string depends on the nature of the bend at the point of plucking. If the string has length L, and the plectrum of infinitely small width is at a point x = a, the string may be represented as in Fig 4.1 .

Figure 4.1

The solution of the wave equation for a finite string is given by equation 1.10 .

$$y = \sum_{n=1}^{\infty} \left\{ A_n \cos \omega_n t + B_n \sin \omega_n t \right\} \sin \frac{n\pi x}{L}$$

At time t = 0 , this reduces to

$$y_0 = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \qquad 4.1$$

The velocity v_0 of the string in the 'y' direction is given by $\partial y/\partial t$ and at t=0,

$$v_0 = \partial y/\partial t = \sum_{n=1}^{\infty} \omega_n B_n \sin \frac{n\pi x}{L} \qquad 4.2$$

Substituting the Fourier Theorem in eqn. 4.1

$$A_n = \frac{2}{L} \int_0^L y_0 \sin \frac{n\pi x}{L} \qquad 4.3$$

Substituting in eqn 4.2

$$B_n = \frac{2}{\omega_n L} \int_0^L v_0 \sin \frac{n\pi x}{L} \qquad 4.4$$

From fig. 4.1, the initial conditions are

$$y = hx/a \qquad a > x > 0$$

$$y = h\frac{(L-x)}{(L-a)} \qquad L > x > a$$

For a plucked string, the string velocity at the moment of release, that is at t = 0 , is zero ; hence $B_n = 0$.

$$A_n = \frac{2}{L} \left\{ \int_0^a \frac{hx}{a} \sin \frac{n\pi x}{L} dx + \int_a^L h.\frac{L-x}{L-a} \sin \frac{n\pi x}{L} dx \right\} \qquad 4.5$$

$$= \frac{2\ h\ L^2}{a n^2 \pi^2 (L-a)} \sin \frac{n\pi a}{L} \qquad 4.6$$

Since A = 0, when $\sin \frac{n\pi a}{L} = 0$, then partials with mode numbers $\frac{L}{n}, \frac{2L}{n}, \frac{3L}{n}$, ... will have zero amplitude. For the harmonic string partials with nodes at the plucking point have zero amplitude.

The amplitude of the remaining partials are proportional to $1/n^2$

4.2.2 Initiation of vibration by striking the string

The analysis of the struck string is quite complex, but is simplified considerably in the following discussion. If the string is hit by a narrow 'knife edge' hammer the string may be represented as in Fig. 4.2 .

$t = o$

$t = t$

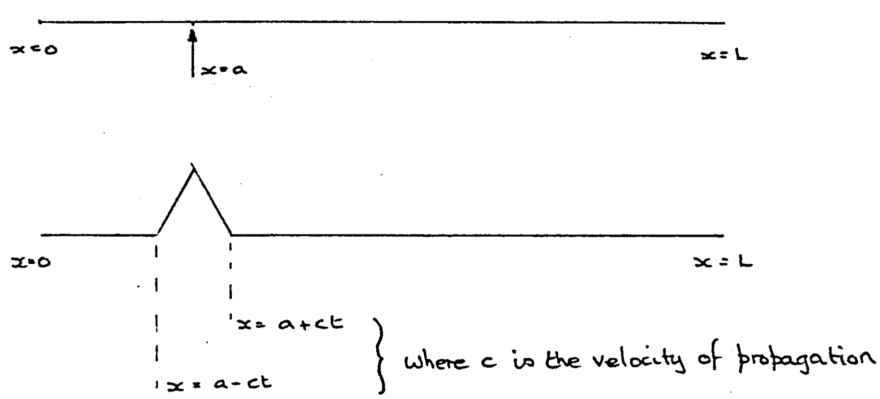


Fig 4.2

For the struck string, equations 4.3 and 4.4 apply.

Since at $t = 0$, the displacement of the string is zero, $A_n = 0$.

Equation 4.4 may be written

$$B_n = \frac{2}{\omega_n L} \int_0^L \frac{dy}{dt} \sin \frac{n\pi x}{L} \, dx \qquad 4.7$$

$$= \frac{2u}{\omega_n L} \sin \frac{n\pi a}{L} \qquad 4.8$$

since the velocity is zero at all x except at a,

and where u is the (velocity of the string at a)$\times$(infinitesimal length where struck).

Since $B_n = 0$ when $\sin n\pi x = 0$, then partials with mode numbers $\frac{L}{n}$, $\frac{2L}{n}$, $\frac{3L}{n}$ . . . . will have zero amplitudes. For the harmonic string partials with nodes at the striking point have zero amplitude.

The amplitude of the remaining partials are proportional to $1/n$.

The difference between the amplitudes of the partials for the plucked and struck strings are shown in fig. 4.3 . The fundamental is shown with an amplitude of 0 dB.and the frequency scale is linear giving an equal distance between the partials.

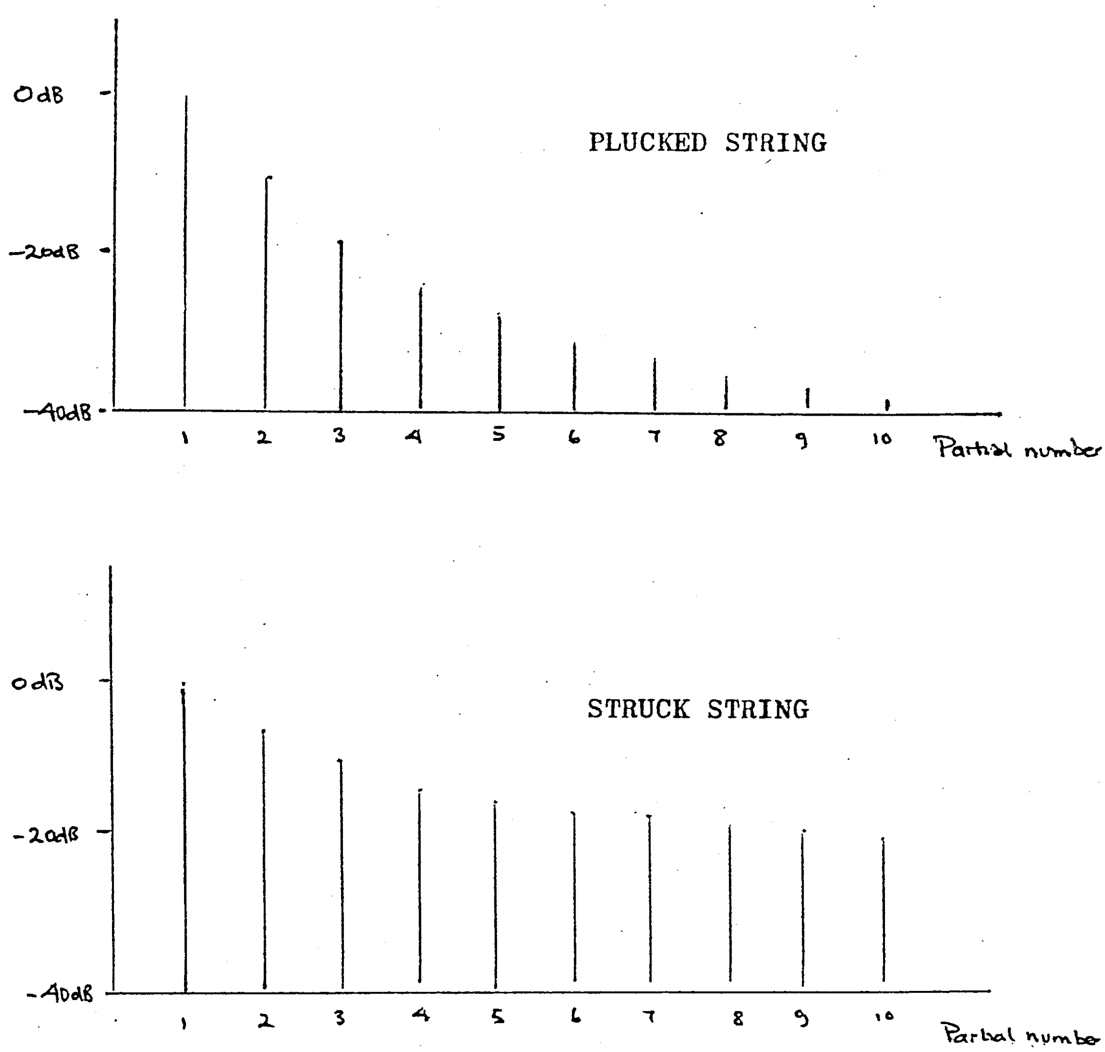


FIG 4.3

It may be plainly seen from the figure above that the struck string is richer in its upper partials than the plucked string.

4.3 Other effects

The preceding discussion described the idealised spectra of piano and harpsichord strings, taking into account the hammer/ plectrum positions. This may be modified by taking into account the effect of piano hammer or harpsichord plectrum length along the string, impact time and string stiffness. These effects are discussed by BENADE (1976), who offers no detailed discussion on how he arrives at his conclusions. The following notes are based on the conclusions given by Benade.

4.3.1 Plectrum position

The principle of supperposition applies when several plectra are used. The spectrum produced by several plectra acting simultaneously may be found by combining the spectrum of each plectrum as if it were acting alone, taking care to account for the initial direction of movement when combining the spectra.

Two plectra pulling in the same direction will act similarly to a plectrum pulled back twice as far, provided the inter-plectrum distance is less that 1/3 of $\lambda_n/2$;
where λ_n is the wavelength of the n^{th} mode measured along the string, and λ_1 is twice the length of the string.

Two plectra pulling in the same direction will produce no excitation for any mode where $\frac{n.\lambda_n}{2}$ = inter-plectrum distance, where n is an odd integer.

For modes where $\lambda_n/2$ is almost equal to the inter-plectrum distance, the excitation will be weak.

4.3.2 Length of the hammer along the string

The idealised force distribution produced by a hammer of length W is shown in fig 4.4 .

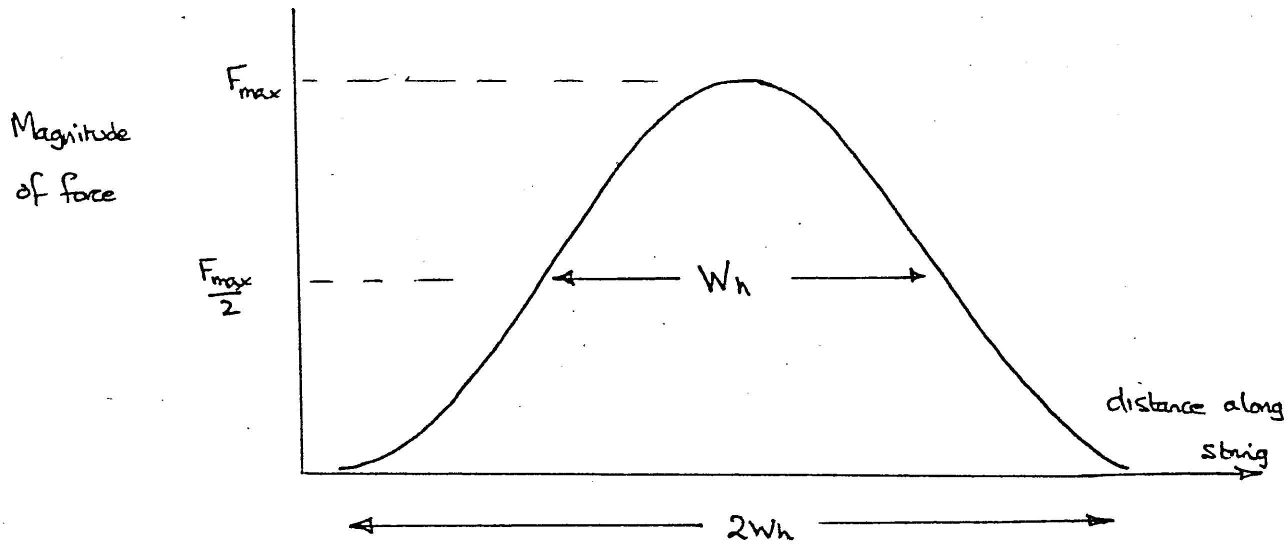


Figure 4.4

The way in which the hammer length modifies the spectrum depends on the relationship between the hammer length and the wavelength of the mode along the string.

Modes for which $W_n < \frac{\lambda_n}{2}$ are excited in almost exactly the same way as by a narrow hammer.

Modes for which $W \simeq \frac{\lambda_n}{2}$ are excited about half as strongly as they would be by a narrow hammer.

Modes for which $W > \lambda_n$ recieve almost no excitation.

4.3.3 Impact time

The idealised force variation with time is shown in fig. 4.5 . The way in which impact time modifies the spectrum depends upon the relationship between contact time T_h and the period of oscillation of the mode under consideration, $P_n = 1/f_n$.

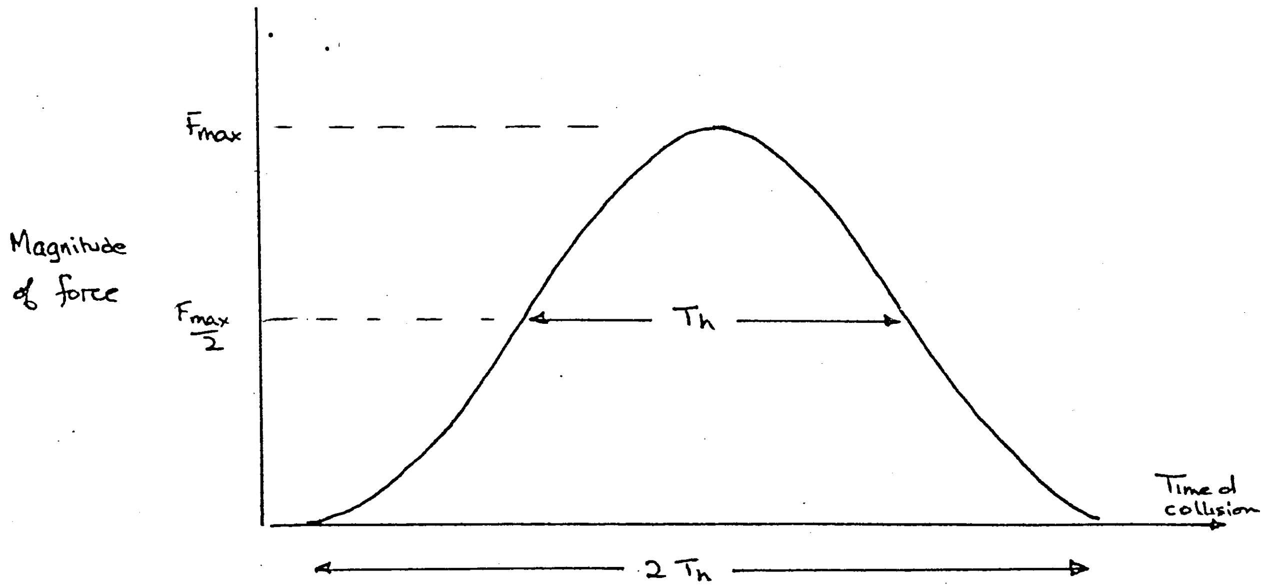


Figure 4.5

Modes for which $T_h < \frac{P_n}{4}$ are excited in almost exactly the same way as by a theoretical hammer which strikes and rebounds instantly.

Modes for which $T_h \simeq \frac{P_n}{2}$ are excited about half as strongly as by a theoretical hammer.

Modes for which $T_h > P_n$ recieve almost no excitation, due mainly to the absorbtive effects of the felt hammer.

4.3.4 String stiffness

In the context of the present discussion string stiffness modifies the initial shape of the string, as shown in fig 4.6 . This effectively lengthens the plectrum or hammer along the string.

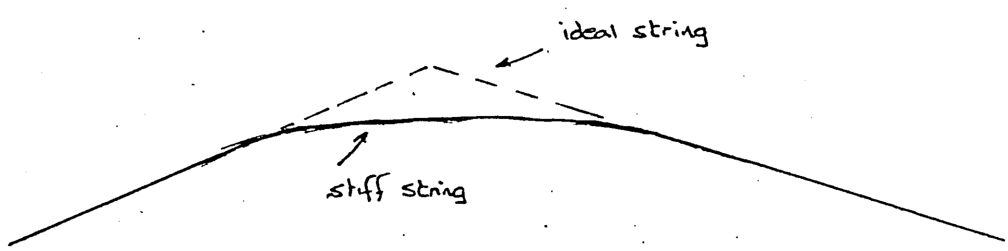


Fig 4.6

4.4.1 General termination and reflection coefficient

For the finite string it is neccessary to consider the complete solution of the wave equation.

$$y = A_1 e^{j(\omega t - kx)} + A_2 e^{j(\omega t + kx)} \qquad 1.5$$

where A_1 = the amplitude of the incident wave

and A_2 = the amplitude of the reflected wave.

At the end of the string the force exerted on the termination is $-T\frac{\partial y}{\partial x}$, which may be found by differentiating eqn. 1.5 .

$$-T\frac{\partial y}{\partial x} = jTk e^{j\omega t}\left\{A_1 e^{-jkx} - A_2 e^{jkx}\right\} \qquad 4.9$$

The velocity at this point is $\frac{\partial y}{\partial t}$ and may be found from eqn 1.5 ,

$$\frac{\partial y}{\partial t} = j\omega e^{j\omega t}\left\{A_1 e^{-jkx} + A_2 e^{jkx}\right\} \qquad 4.10$$

The ratio of the force at the termination to the velocity at the termination is the terminating impedance Z_T .

Hence $$Z_T = \frac{-T\frac{\partial y}{\partial x}}{\partial y/\partial t} = \frac{Tk}{\omega}\left\{\frac{A_1 e^{-jkx} - A_2 e^{jkx}}{A_1 e^{-jkx} + A_2 e^{jkx}}\right\} \qquad 4.11$$

Since $Z_o = Tk/\omega$

$$Z_T = Z_o\left\{\frac{A_1 e^{-jkx} - A_2 e^{jkx}}{A_1 e^{-jkx} + A_2 e^{jkx}}\right\} \qquad 4.12$$

At the origin, where x = 0,

$$Z_T = Z_o\left\{\frac{A_1 - A_2}{A_1 + A_2}\right\} \qquad 4.13$$

Letting $K = A_2/A_1$, the complex amplitude reflection coefficient

$$K = \frac{A_2}{A_1} = \frac{Z_o - Z_T}{Z_T + Z_o} \qquad 4.14$$

and $$\frac{Z_T}{Z_o} = \frac{1-K}{1+K} \qquad 4.15$$

Since power is proportional to amplitude squared,

the power reflection coefficient = $\alpha_R = |K|^2$ 4.16

and the power transmission coefficient = $\alpha_T = 1 - \alpha_R$

$$= 1 - |K|^2 \qquad 4.17$$

4.4.2 The string in damped resonant vibration

The solution of the wave equation in section 1.3 assumed rigid supports, and the implicit assumption of no damping. Damping may be caused by

a) loss of energy to the air by radiation

b) losses caused by internal friction of the string

c) loss of energy to the supports.

In the following only the latter will be discussed.

From section 1.6, the energy associated with a vibrating string is

$$E = \frac{\omega^2 A^2 M}{4} \qquad (1.16)$$

Since the amplitude of vibration is related to the velocity amplitude by $V^2 = \omega^2 A^2$, where V is the velocity amplitude, eqn. 1.6 may be written

$$E = \tfrac{1}{4} M V^2$$

The energy E associated with the string will vary slowly with time as energy is lost.

The power flow to each support $= \tfrac{1}{2} Z_0 \left(\frac{V}{2}\right)^2 = \frac{1}{8} Z_0 V^2$

$$= \frac{1}{8} \mu c V^2$$

The rate of energy lost to EACH support $= -\partial E/\partial t$

$$= 1/8 \ \mu c \alpha_T V^2$$

For both supports, the rate of loss of energy from the STRING

$$-\partial E/\partial t = -1/4 \mu c \alpha_T V^2$$

$$= -\frac{\alpha_T c E}{L}$$

The decay rate of energy $= \dfrac{\partial E}{\partial t} = \alpha_T c/L \qquad (4.18)$

If the initial energy at time t=0, is E_0

The energy at some time later is given by

$$E = E_0 \, e^{-\frac{\alpha_T c}{L} t}$$

$$\ln E = \ln E_0 + t \left\{ -\frac{\alpha_T c}{L} \right\}$$

If the reverberation time is defined as the time taken for the energy to fall to 10^{-6} of its initial value (a fall of 60 dB)

Then the reverberation time = $T_{60} = \dfrac{L}{\alpha_T . c.} \ln 10^{-6}$ (4.19)

The assumption that the impedance of the supports is much greater than the characteristic impedance of the string is supported by FUCHS (1976), who give for the piano a figure of 600 ; 1 .

Equation 4.19 agrees with that given by BENADE (1976) who gives

$$\text{decay time} \propto \frac{\text{moving mass}}{\alpha_T}$$

4.5.1 Measurement of decay of piano tones

In the following experiment the main objective was to examine the effect of triple and single stringing on piano decay times. As a piano was only available for a short time it was decided to only examine a single note in each octave.

The piano used in the experiments was that used in the previous experiments, described in section 2.7.2 .

The piano tones were first recorded as they are on the piano, and then the recording repeated with each piano note having only one string sounding, the other strings being muted with felt wedges.

The notes to be examined were tuned and the action regulated by an experienced piano technician. A B&K 1" microphone type 4145 was placed 1 metre behind the soundboard and the signal fed to the amplifier section of a B&K 2010 analyser. The tones were recorded on a series 3 Nagra tape recorder at a speed of 381 mm/s. The notes were struck by hand with approximately equal force, a mechanical striker being unavailable.

The stringing on the piano was as follows

C1 C2	Single strings	wound
C3	Bi-chord	wound
C4 C5 C6 C7 C8	Tri-chord	plain steel

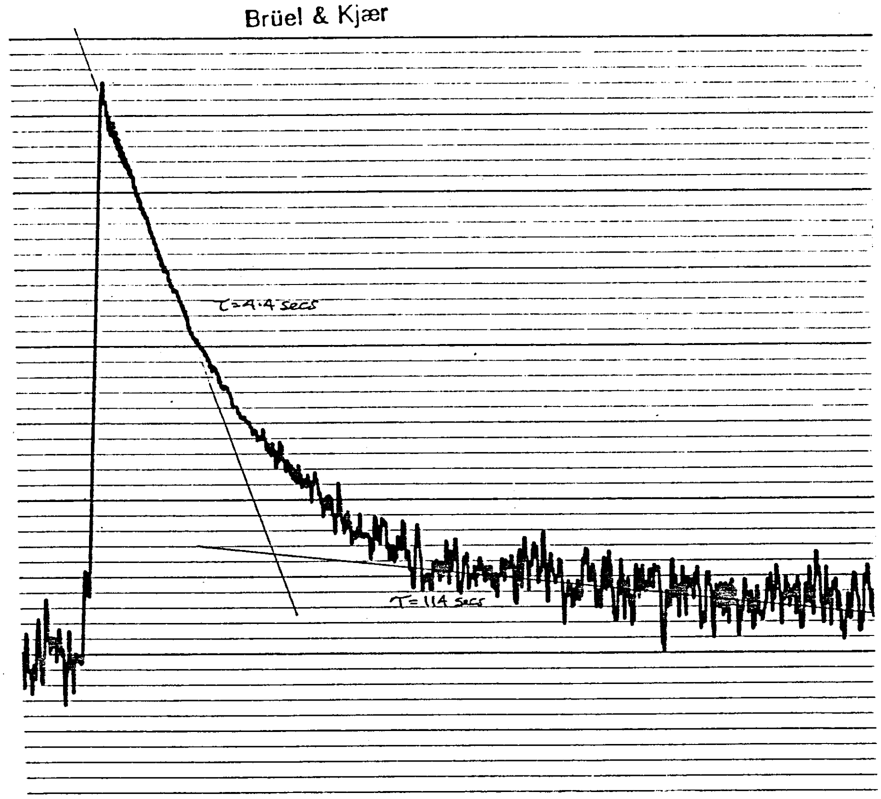
The tape was replayed through a B&K analyser type 2010 in the non-selective mode, and the decay curves plotted on a B&K level recorder type 2307.

Paper speed	10 mm/s
Writing speed	500 mm/s
Potentiometer	50 dB
Lower limiting frequency	20 Hz
Rectifier	True R.M.S.

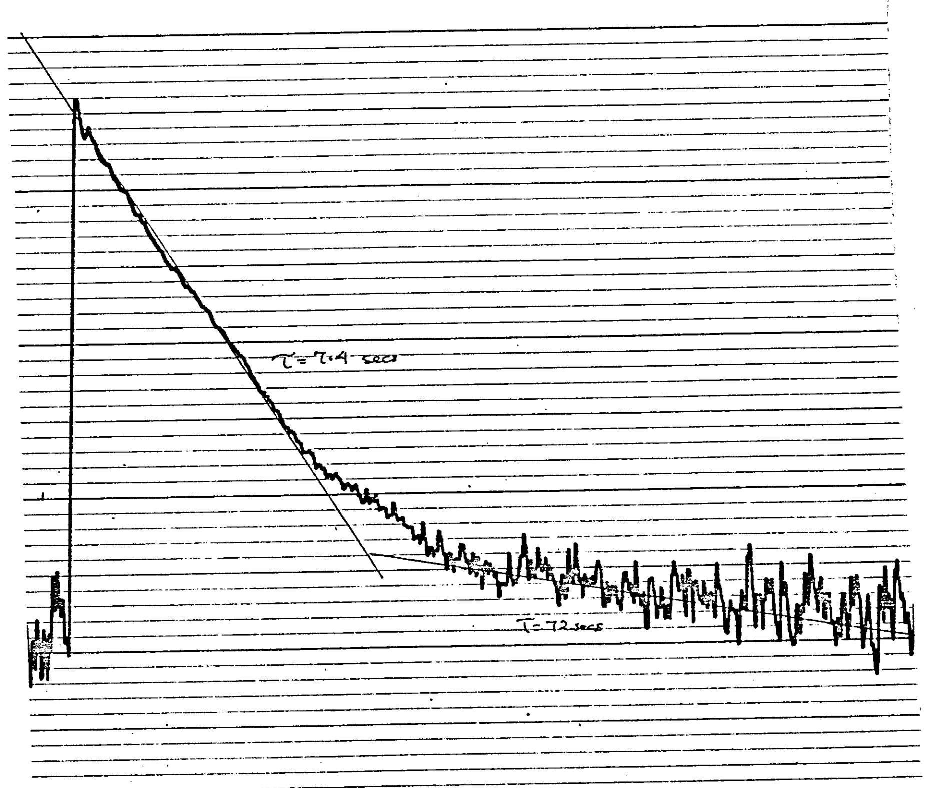
The reverberation times for both multiple strings and single strings were measured using the protractor supplied with the level recorder. The resulting graphs from the level recorder are shown on pages 4.13 to 4.17.

Piano tones - as on the piano.

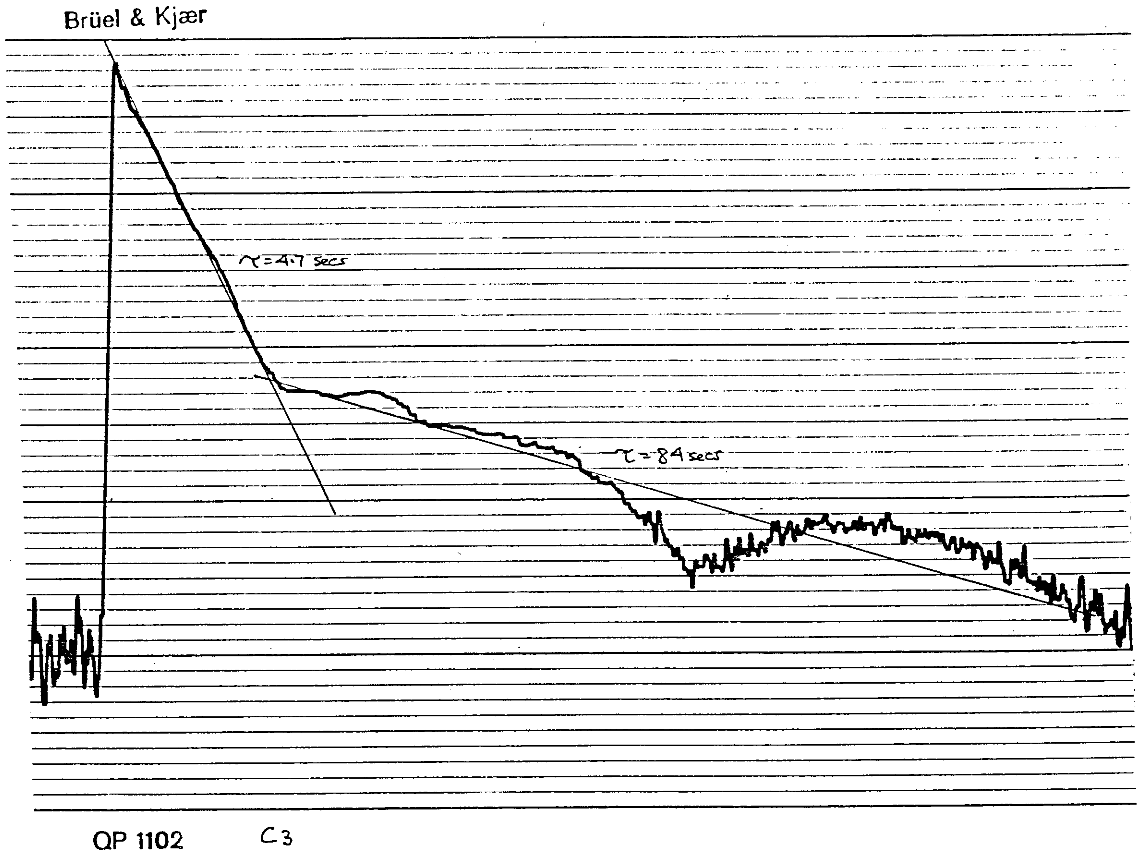
Brüel & Kjær

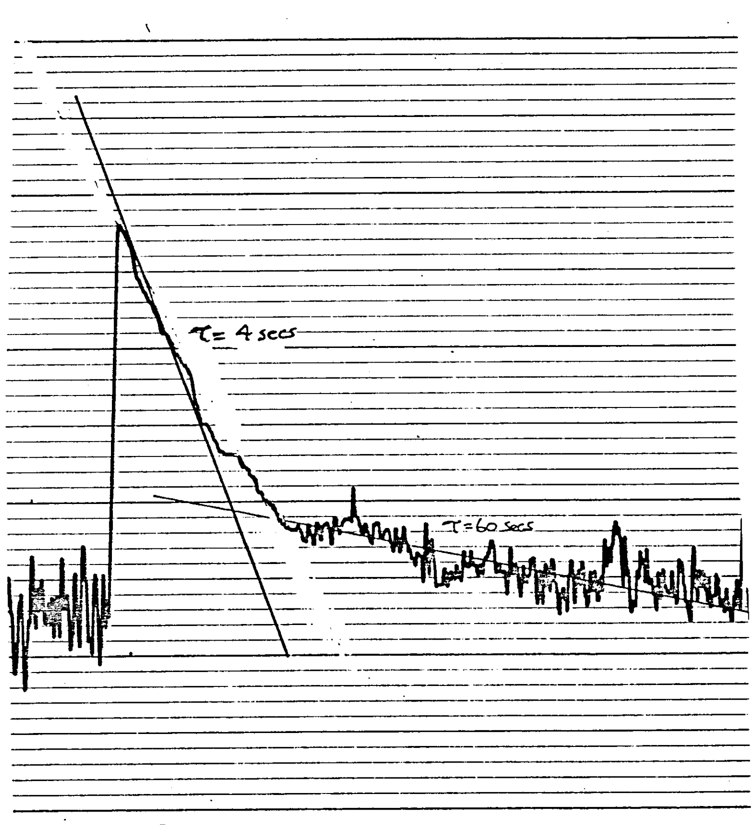


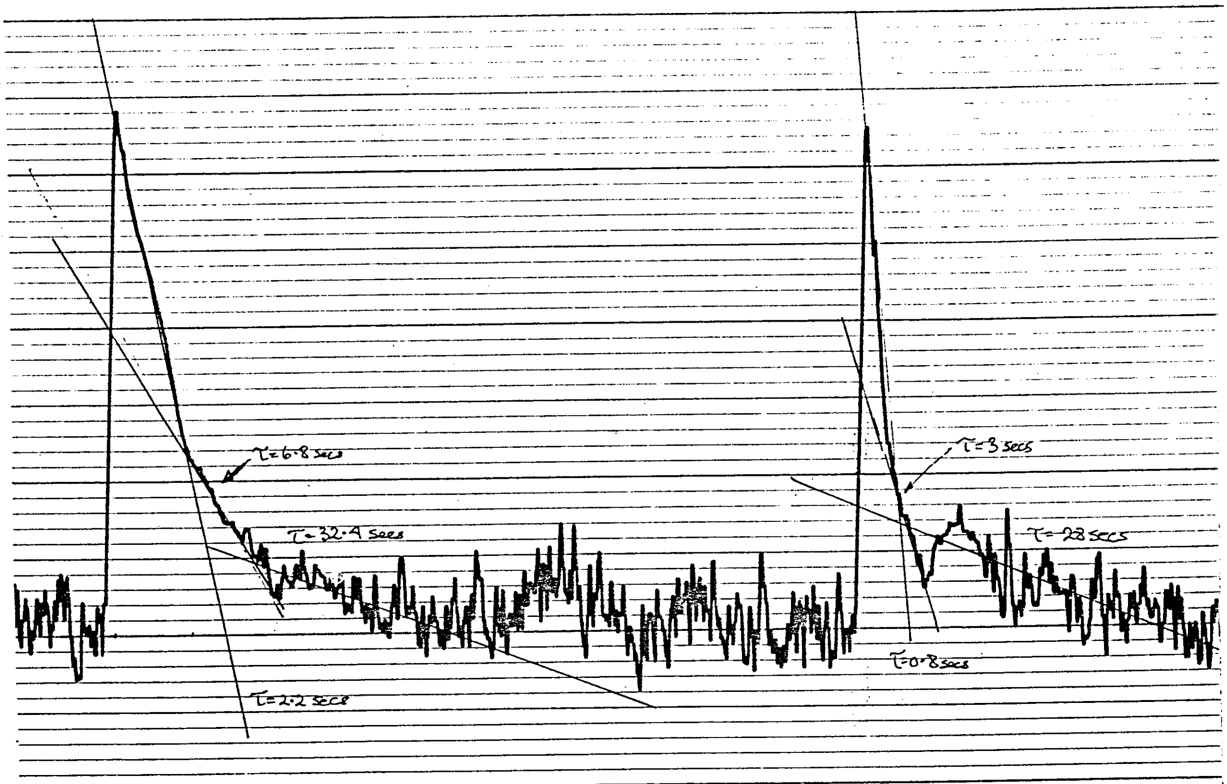
C_1 QP 1102



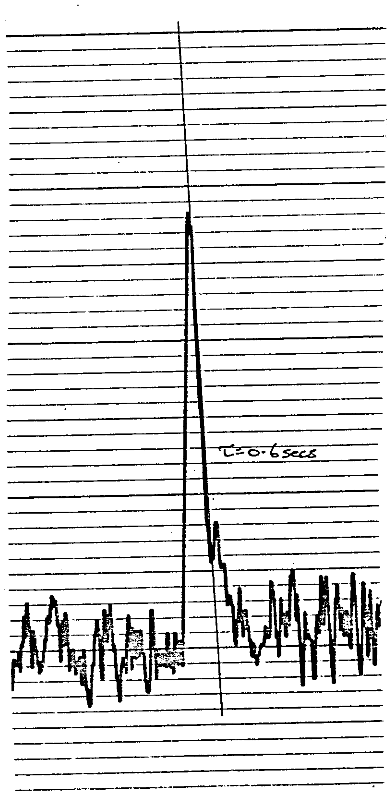
C_2


Brüel & Kjær
τ=4·7 secs
τ=84 secs
QP 1102
C3

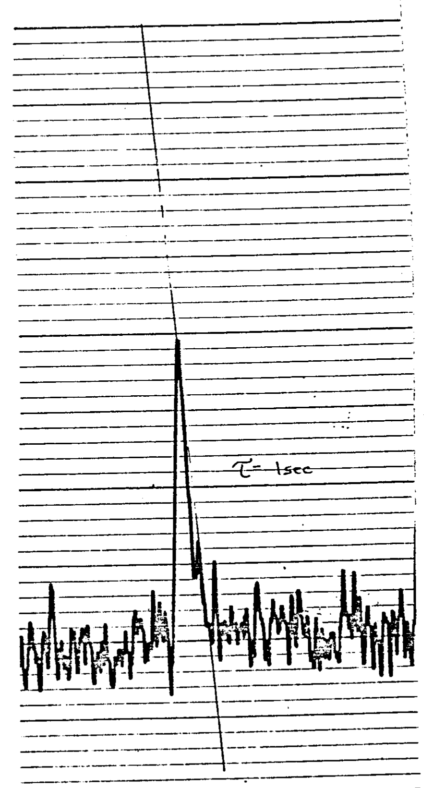

τ= 4 secs
τ=60 secs
C4


τ=6·8 secs
τ=32·4 secs
τ=2·2 secs
τ=3 secs
τ=28 secs
τ=0·8 secs

C5 C6

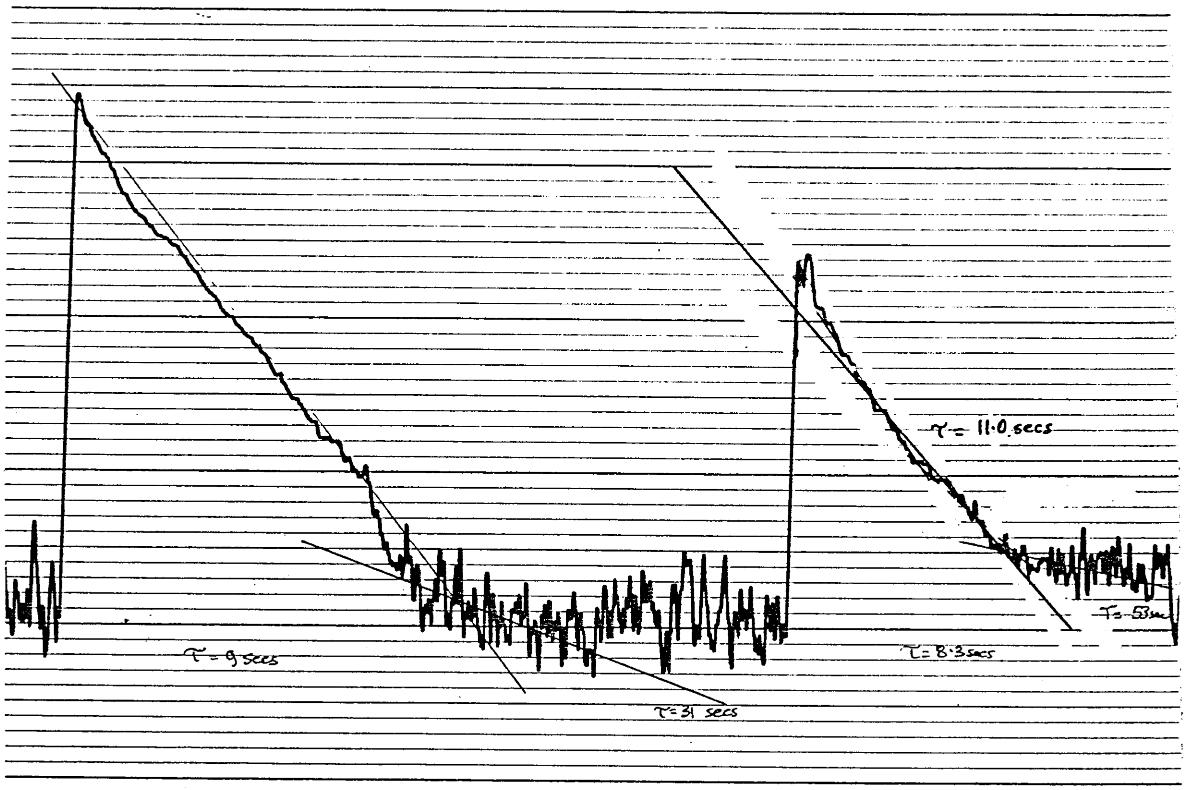

τ=0·6 secs

C7


τ= 1sec

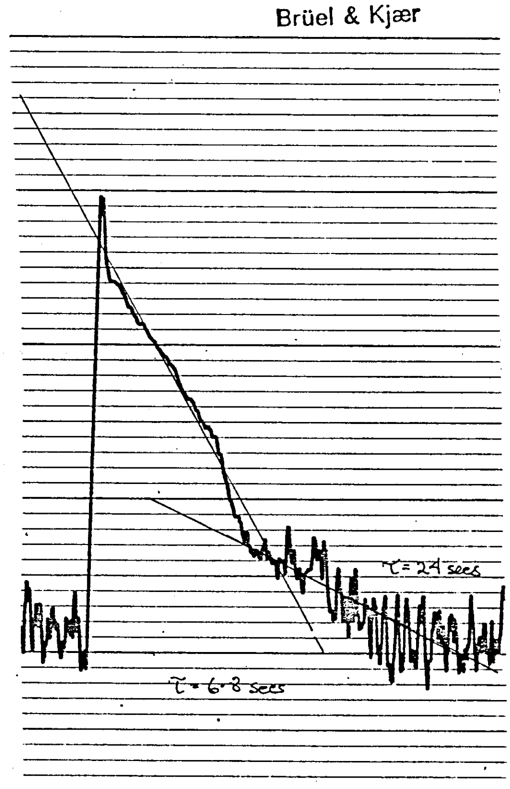
C8

Piano tones - single strings only - other strings damped

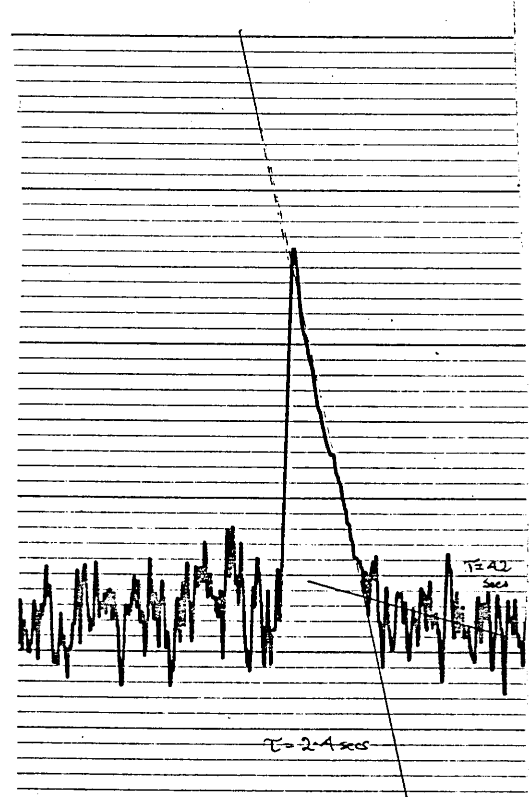


C3 C4

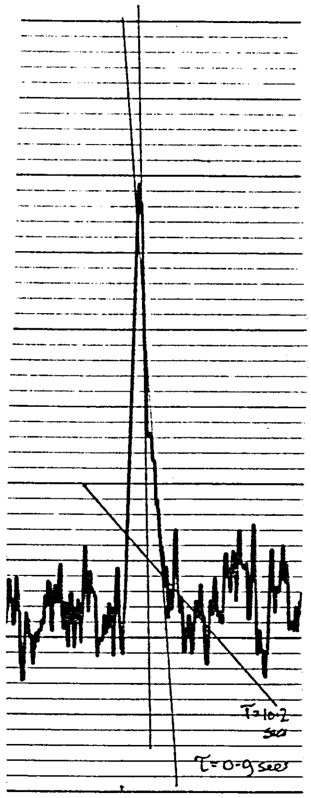
Brüel & Kjær



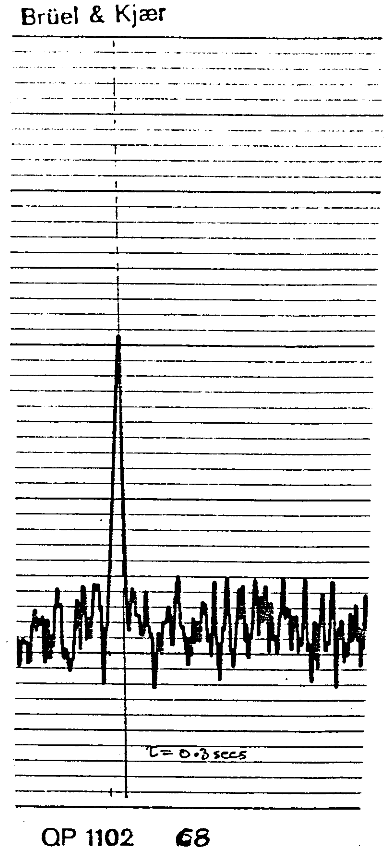
C5 QP 1102



C6



C7



From the decay curves shown on pages 4.13 to 4.17, the measured reverberation times were as follows;

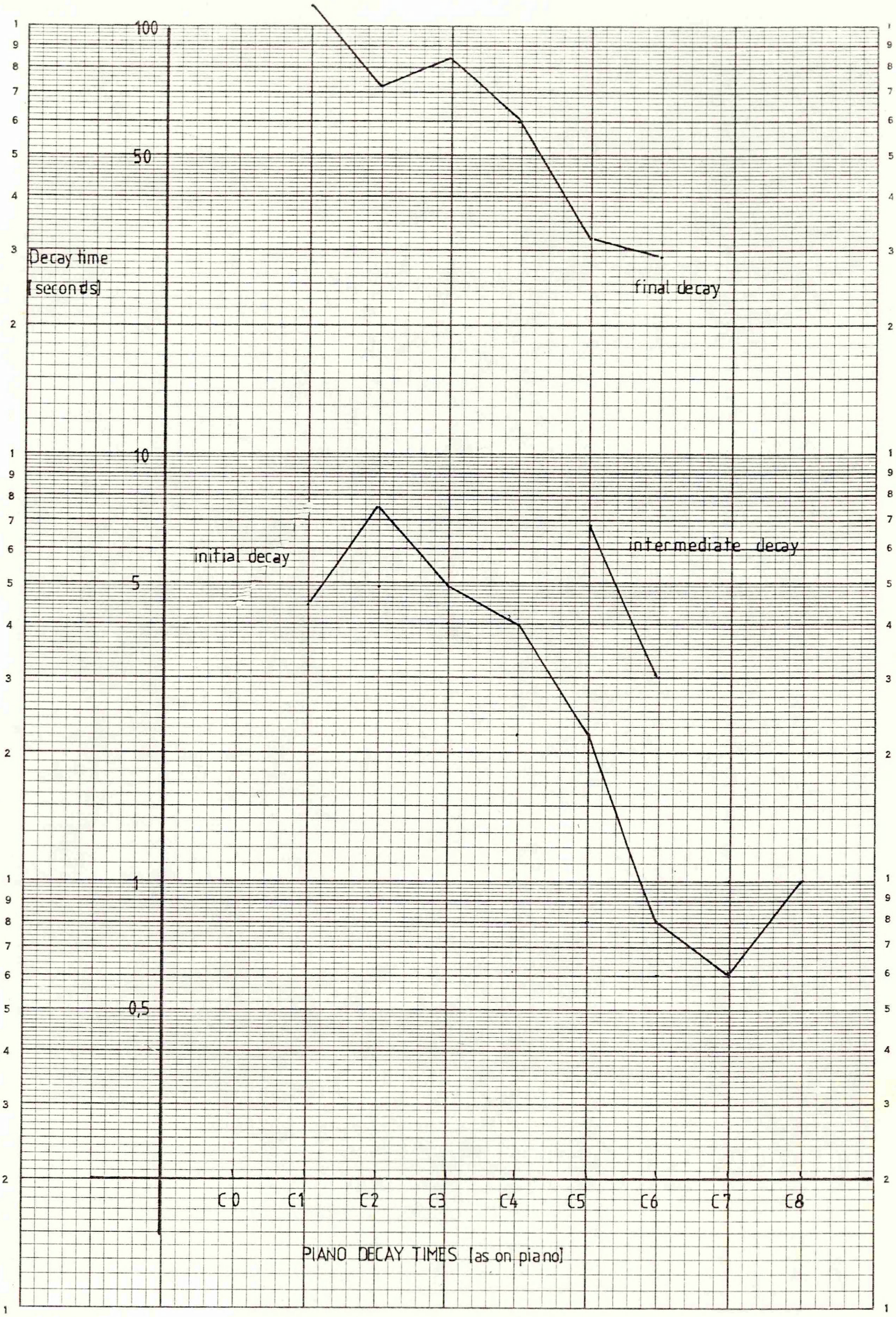
Reverberation times

	As on the piano			Single strings	
	Initial	Intermediate	Final	Initial	Final
C1	4.4		114		
C2	7.4		72		
C3	4.7		84	9.0	31
C4	4.0		60	11	53
C5	2.2	6.8	32	6.8	24
C6	0.8	3.0	28	2.4	42
C7	0.6			0.9	10
C8	1.0			0.3	

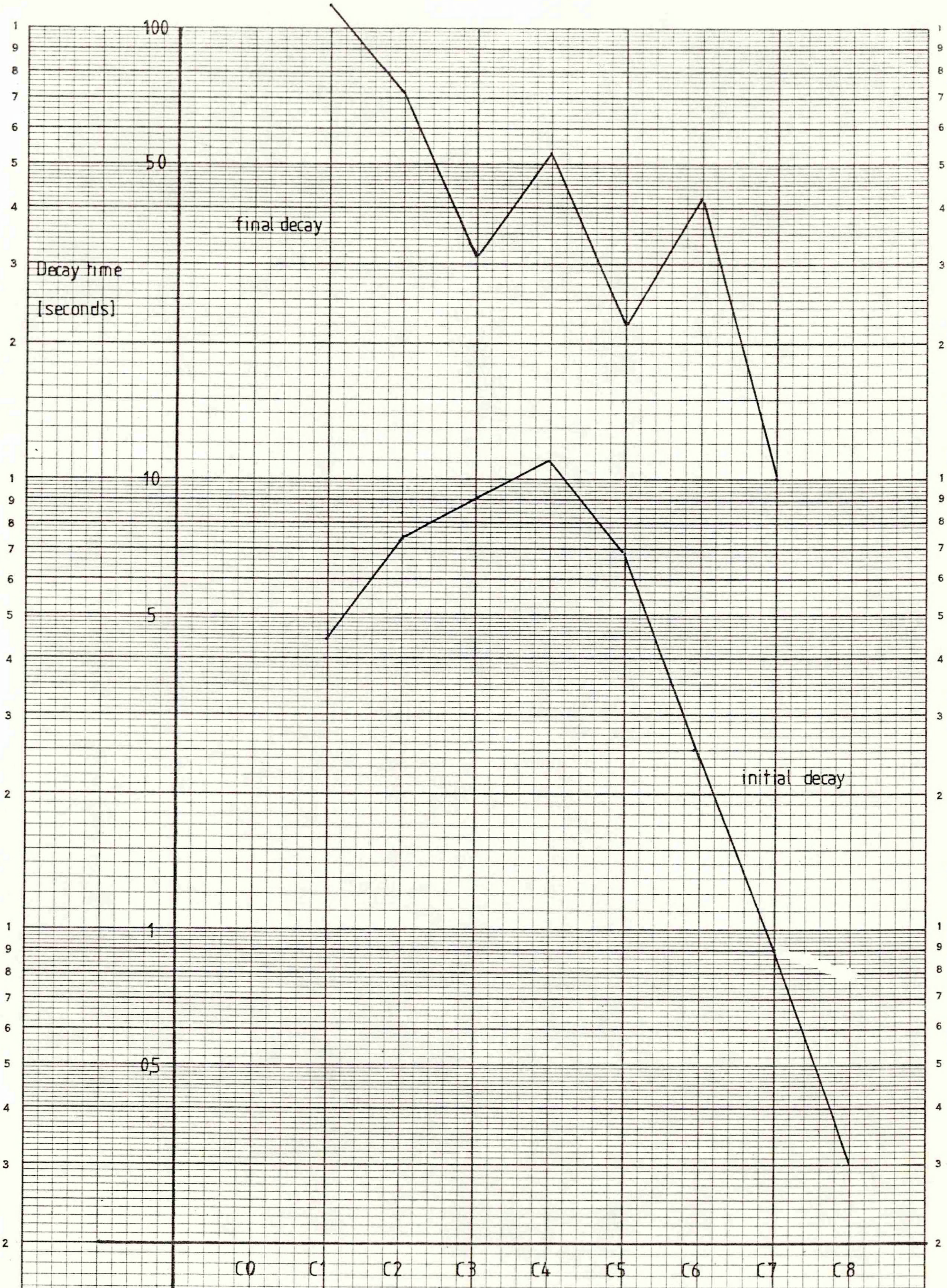
Graphs of the above reverberation times were plotted, and these are shown on pages 4.19 and 4.20 .

For the bi-chords, the ratio of initial single string reverberation time to initial bi-chord reverberation time was approximately 2;1 .

For the longer string length tri-chords, the ratio of initial string reverberation time to initial tri-chord reverberation time was approximately 3;1 . This did not hold for the very short strings.



Decay time
[seconds]
100
50
10
5
1
0,5
final decay
initial decay
intermediate decay
C0
C1
C2
C3
C4
C5
C6
C7
C8
PIANO DECAY TIMES [as on piano]



PIANO DECAY TIMES [single strings]

4.5.2 Comparison of the author's results with other experimenters' work

The reverberation times measured correspond closely with the results obtained by MARTIN (1947). He concluded that a double reverberation time is typical for the lower four octaves of the piano, and gives initial reverberation times varying from 10 seconds in the bass to just under 1 second in the extreme treble. His graphs show a smoother change in initial reverberation times for a baby grand piano compared with his results for an upright piano.

In the authors' experiments, the 3:1 ratio for initial reverberation time with single strings compared with the same strings as tri-chords confirms the theory given by BENADE (1976). Benade states that when initially excited the three strings of the tri-chord vibrate in phase and hence the wave impedance of a tri-chord is three times that of a single string; when the strings become out of phase due to tuning differences the impedance falls and the reverberation time increases.

In the author's experiments, the intermediate reverberation time is approximately three times the initial reverberation time, and the same as the single string reverberation time as may be seen for the intermediate reverberation times noted for the C5 and C6 strings. In the author's experience another phenomena occasionally occurs; sometimes a tri-chord may be tuned too perfectly and the strings of the tri-chord continue to vibrate in phase, giving a very loud note with a short reverberation time. This gives a reverberation time which does not match those of the adjacent notes, which is musically undesirable. When this occurs most tuners remedy the situation by a slight de-tuning.

FLETCHER (1976) notes the other factors which affect decay. He notes that the other two main causes are

1) Air damping $\propto \rho r^2$ which holds for $f \ll r^{-2}$

$\propto \rho r^2 f^{-2}$ which holds for $f \gg r^{-2}$

2) Internal damping $\propto \frac{1}{f} \frac{Y_1}{Y_2}$ which is negligible for steel and piano wire.

where ρ = density

$Y = Y_1 + j Y_2$, complex Young's modulus

r = radius of the string

f = frequency of vibration

In a later paper FLETCHER (1977) gives more detail of his theory and also some experimental results relating to the harpsichord, unfortunately his experimental reverberation times are "estimated decay times to inaudability".

4.5.3 Comparison of piano decay and similar experiments with a harpsichord

A survey of the literature yielded little information on harpsichord decay times. As stated above FLETCHER (1977) only gives "estimated decay times to inaudability".

From the previous experiment on piano decay it would be expected that the double decay found on multiple strings activated by a single hammer would not be found if multiple strings were activated by separate plectra as in the harpsichord. If this were the case it would suggest that the double decay for piano strings is due to strings vibrating in phase and later at random phase. Strings activated by separate plectra would initially vibrate with random phase, and hence have only a single decay.

The harpsichord used in the following experiments was made at the London College of Furniture, being a copy of a Ruckers, having two 8' stops one fitted with a leather buff, and also a single 4' stop.

The harpsichord was tuned and voiced by an experienced harpsichord technician.

A B&K 1" microphone type 4145 was placed 1 metre above the soundboard and the signal fed to the amplifier section of a B&K 2010 analyser. The tones were recorded on a series 3 Nagra tape recorder at a tape speed of 381 mm/s. The notes were struck by hand with approximately equal force, a mechanical striker being unavailable.

The harpsichord tones recorded were

single 8'

double 8'

single 8' with buff stop.

The 4' stop was not recorded. Strings not being played were damped by the felts on the jacks.

The tape was replayed through the B&K 2010 analyser in the non-selective mode, and the decay curves plotted on a B&K level recorder type 2307.

Paper speed	10 mm/s
Writing speed	500 mm/s
Potentiometer	50 dB
Lower limiting frequency	20 Hz
Rectifier	True R.M.S.

The resulting graphs of harpsichord decay curves are shown on pages 4.25 to 4.27 .

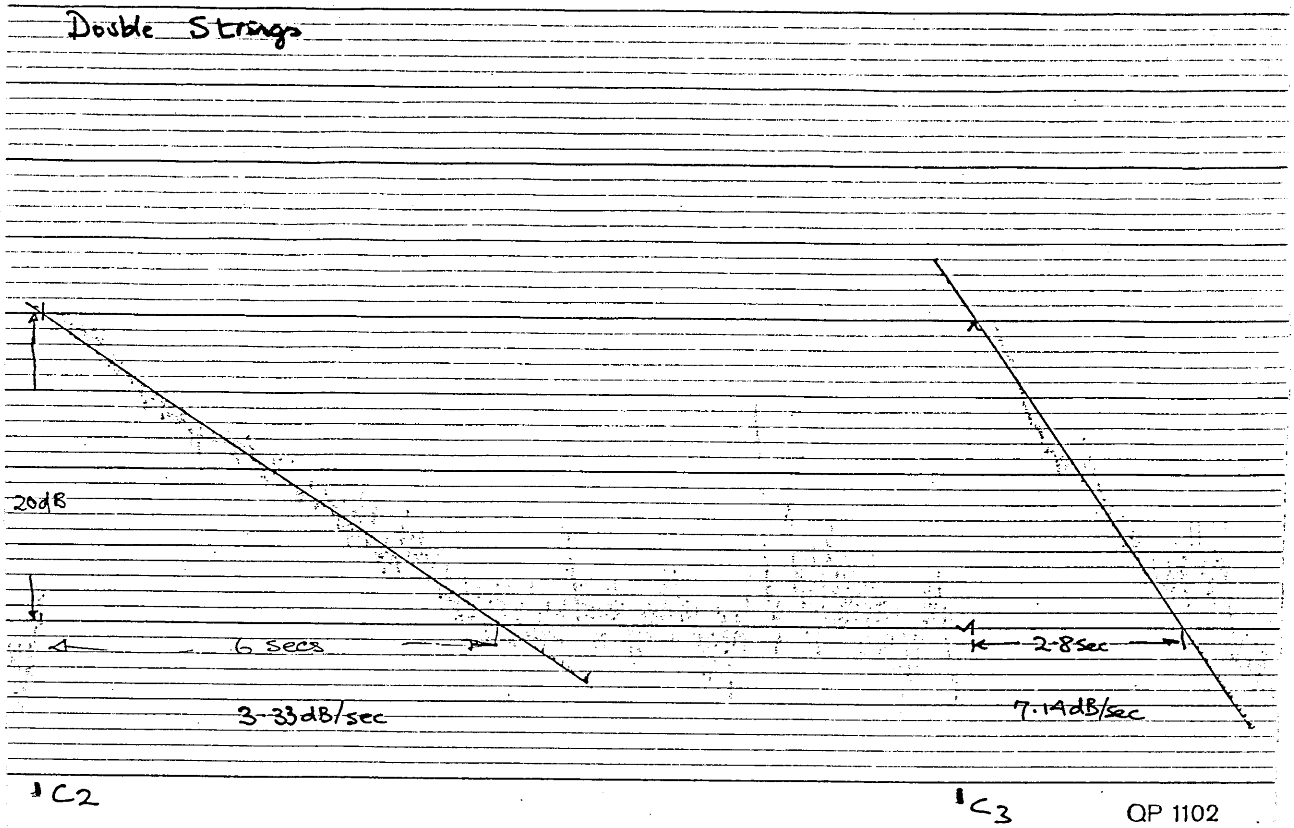
The reverberation times were measured using the protractor supplied with the level recorder and are tabulated below.

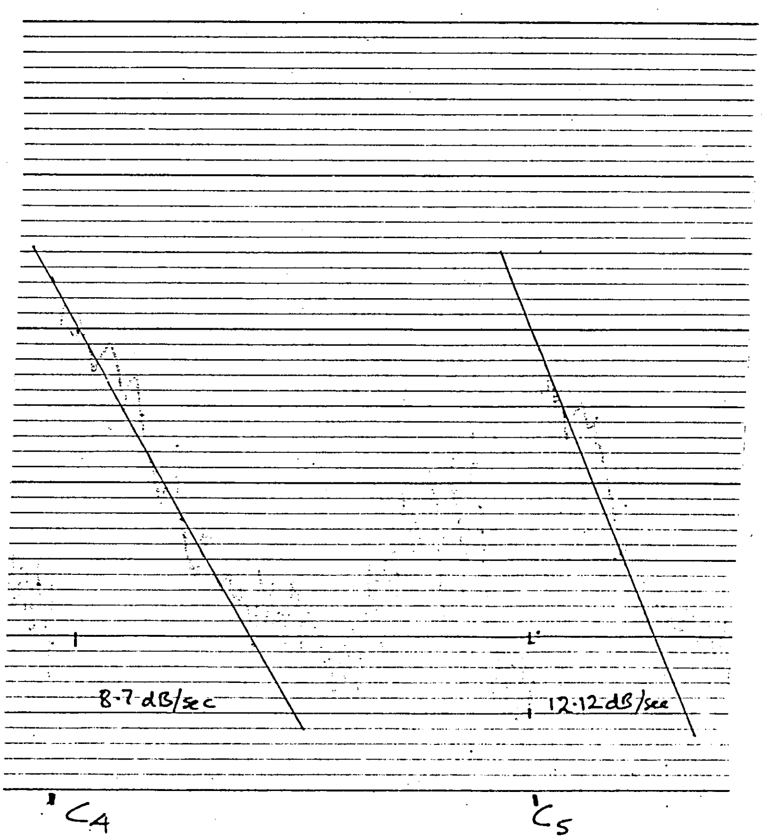
Note	String material	Length	Diameter	Reverberation time		
		mm	mm	seconds		
				2 x 8'	1 x 8'	1 x 8' buff
C2	Brass	1555	0.5	18	16.2	12.8
C3	Phosphor bronze	1100	0.3	8.4	11.4	6.3
C4	Steel	670	0.27	6.9	5.4	2.3
C5	Steel	330	0.25	4.9	3.9	1.4
C6	Steel	170	0.2	2.5	3.0	0.9

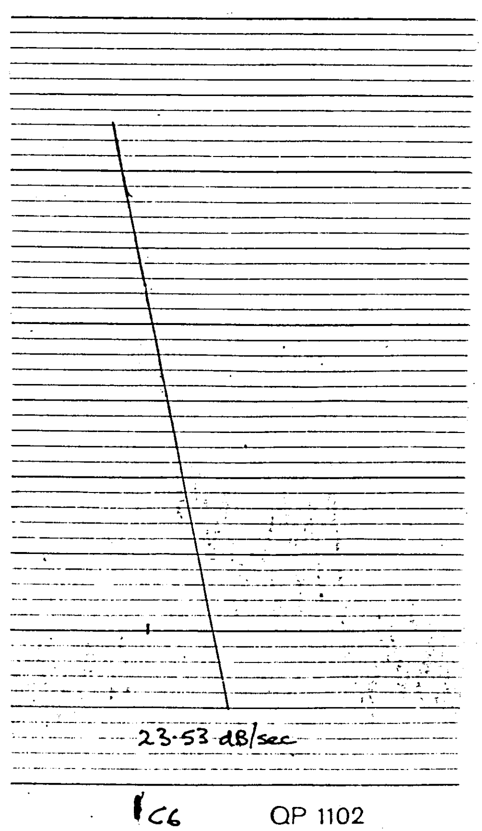
From the above table graphs of reverberation time against note were drawn, and these may be seen on page 4.28 .

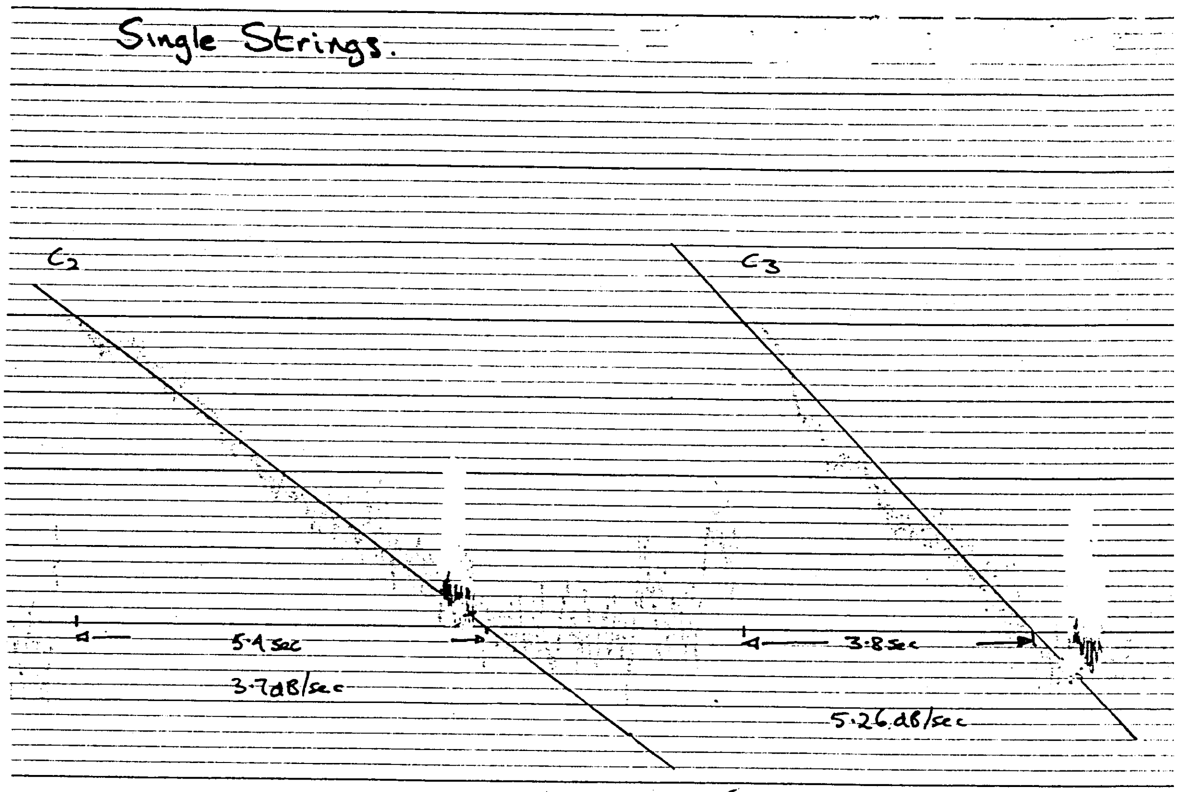
The results are very much as predicted, similar reverberation times for single and double strings and a much shorter reverberation time for the buff stop.

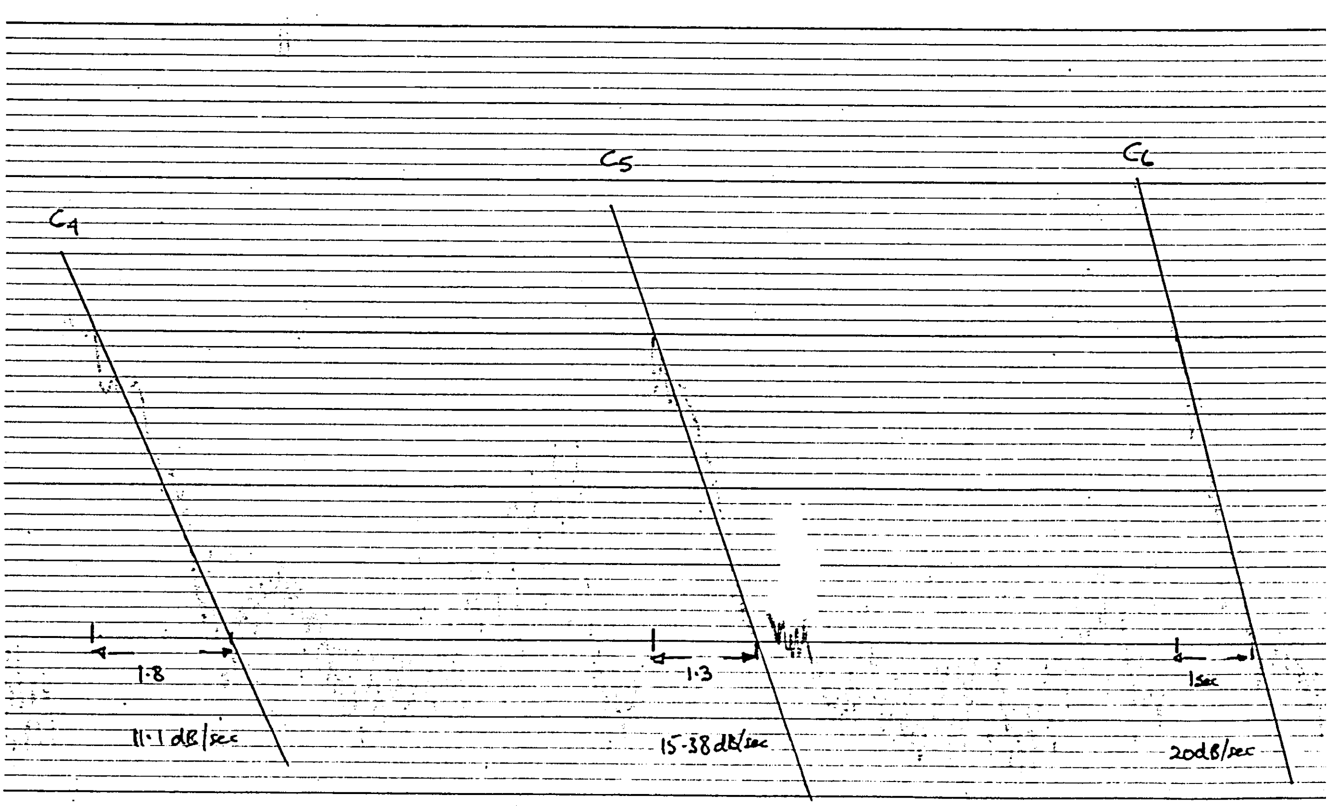
As with the piano, the higher strings have shorter reverberation times. Unlike the piano, multiple strings do not exhibit a double decay. This is because with the piano multiple strings hit by the same hammer initially vibrate in phase, the later slow decay occuring when the strings have started to vibrate out of phase. Since the harpsichord used separate plectra for each string in multiple strung notes the strings initially vibrate with random phase, accounting for their single decay time.


Double Strings
20dB
6 secs
3·33dB/sec
2·8sec
7·14dB/sec
C2
C3
QP 1102

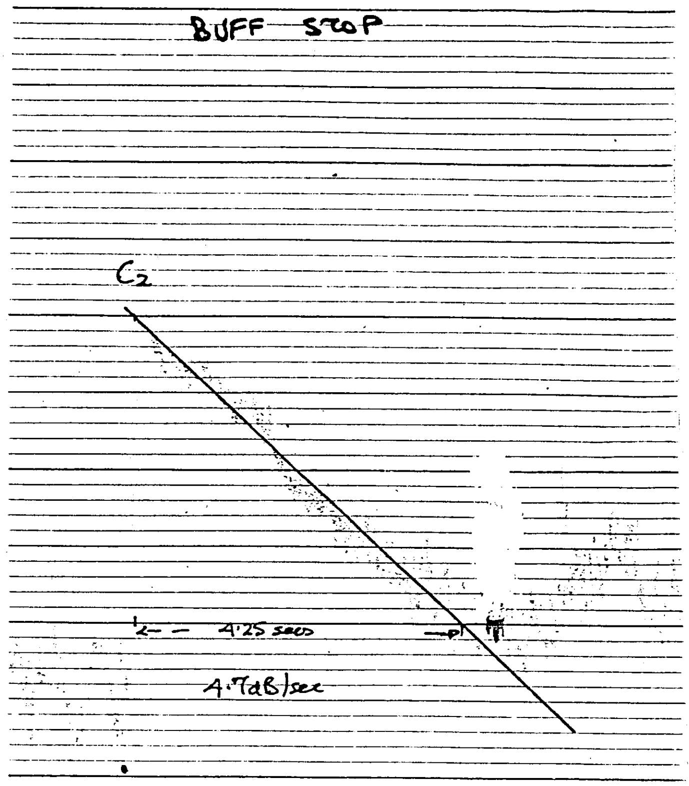

8·7 dB/sec
12·12 dB/sec
C4
C5


23·53 dB/sec
C6
QP 1102

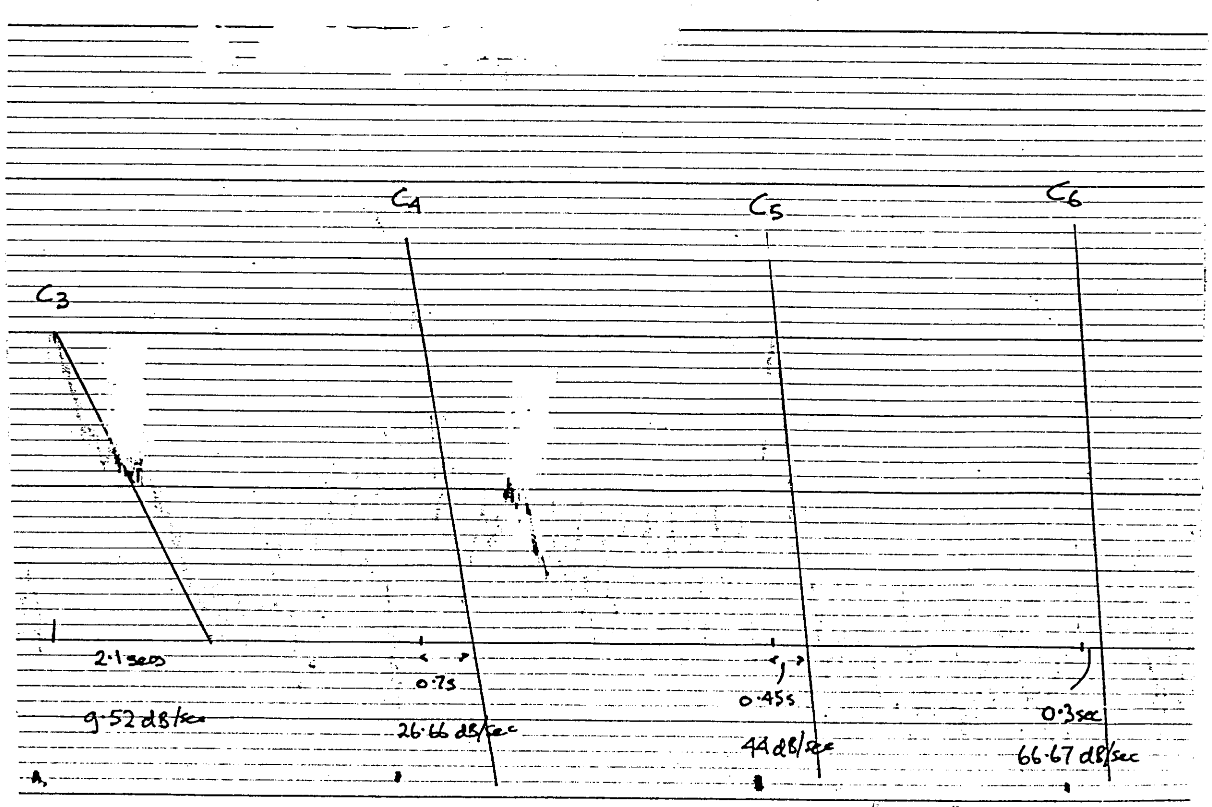

Single Strings.
C2
C3
5.4 sec
3.7 dB/sec
3.8 sec
5.26 dB/sec

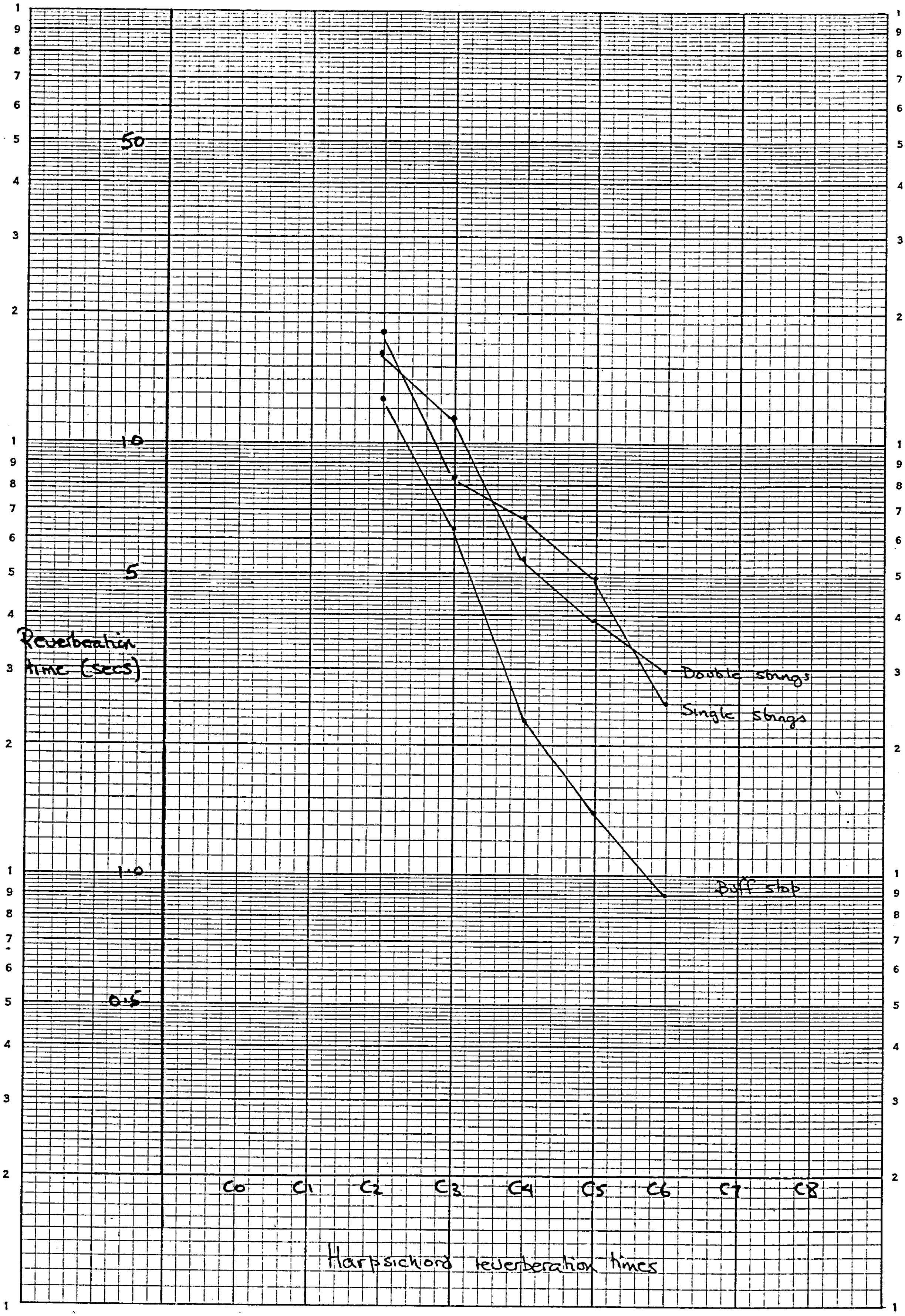

C4
C5
C6
1.8
11.1 dB/sec
1.3
15.38 dB/sec
1 sec
20 dB/sec

Harpsichord decay - buff stop



QP 1102




50
10
5
1.0
0.5
Reverberation time (secs)
Double strings
Single strings
Buff stop
C0
C1
C2
C3
C4
C5
C6
C7
C8
Harpsichord reverberation times

5 Summary and conclusions

The simple treatment of the vibrating string outlined in the first chapter does not adequately describe the vibrating string as found in the piano. The main factor affecting the string which was not taken into account was the stiffness of the string. It was shown in the second chapter that the main effect of stiffness was a raising of the partial frequencies from the frequencies predicted by the simpler treatment. These results were confirmed experimentally in the third section dealing with inharmonicity.

The major effect of stiffness and the consequent effect on the partial frequencies of the strings is the effect on the tuning of the piano. Using conventional tuning methods this results in the middle (or temperament) octave not being tuned exactly to equal temperament. The upper end of this octave is approximately 2.2 cents sharper than the lower end, the exact figure depending on the scaling (string length and diameter) of the piano. The deviations are shown for the piano used in the experiments on page 3.16. The effect will be continued to the strings outside the middle octave, and results in the tuning having "stretched octaves", that is the higher octaves become progressively sharper and the lower octaves progressively flatter.

The middle octave could be tuned exactly to equal temperament if the beat rates calculated in section 3.3 were used. This would result in a beating octave from the lower end of the tuning octave to the upper end of the tuning octave. The beat rates to achieve this tuning however are dependant on the scaling of the particular piano being tuned, and hence is impracticable as the beat rates would have to be re-calculated for each different model of piano.

The stiffness of the strings also affects the initial shape of the plucked or struck string. This smoothing of the initial shape of the string before it begins to vibrate will change the partial structure, resulting in a lowering of the amplitude of the higher partial in the spectrum. The idealised spectra for plucked and struck strings are discussed at the beginning of section four.

Another difference between the ideal string and the piano string is that the supports (or bridges) on the piano do move and are not stationary. A brief discussion at the end of the first chapter concludes that this could result in a raising or lowering of the partial frequency depending on the behaviour of the soundboard and bridge at these particular frequencies. This is shown diagrammatically in figure 1.4.

The decay of both (struck) piano and (plucked) harpsichord strings are discussed in section four. The experimental results obtained for the piano string agreed well with results obtained by previous experimenters. Both single and multiple course string reverberation times were measured and the initial reverberation time to intermediate time was found to have the expected 3;1 ratio. The final reverberation time measured was probably due to the acoustics of the room in which the experiments were performed.

For the harpsichord, multiple stringing did not significantly affect the reverberation times, which was expected. The buff stop reduced the reverberation time to 80% in the bass and 30% in the treble.

There is scope for further work on the acoustics of the piano and harpsichord. In particular the relationship between string and soundboard impedance could be investigated with regard to the effects on the partial frequencies and also the reverberation times.

The reverberation time experiments could be repeated in an anechoic chamber to remove the effects of room acoustics. Alternatively an electro-magnetic pick up or an accelerometer may be used if an anechoic chamber is unavailable.

An important conclusion from the experiments and survey of the theoretical literature is that the scaling (i.e. string lengths and diameters) of an instrument are of major importance. They have an affect on the reverberation time as noted in section 4.5.2 but more important affect the tuning of the instrument. Although the conclusions are mainly of interest to the designer they are also important to the musician. In the performance of music requiring two keyboard instruments compatible tuning is desirable. Hence both instruments should have the same scaling i.e they should not only be of the same manufacture, but be of the same model.

APPENDIX A1

PRACTICAL VERSIONS OF MERSENNES LAW

The notes in this appendix are limited to versions used in Britain. Practices in the U.S.A. may be found in McFERRIN (1972). The following forms of Mersennes Law are practical in the sense that they are those used in the British piano industry. The use of a constant K is not strictly correct, as this constant does vary, and is not dimensionless.

Mersennes Law may be written

$$f = \frac{1}{2L}\sqrt{T/\rho A}$$

where A is the cross section area of the string

$$A = \pi d^2/4$$

Hence

$$\frac{(fLd)^2}{T} = \frac{1}{\rho\pi} = K$$

It is common practice in the piano industry to express tensions by the mass which when acted upon by gravity produces that tension. For piano wire, ρ , density is 7800 kg.m.^{3}, and if the length and diameter of the wire are measured in millimetres. constant K becomes 400 x 10^6.
This version of Mersennes Law has been recently used in City and Guilds examinations for piano technicians. (Stringed Keyboard Instrument Design, C&G subject no. 563'76)

The same equation $K = \frac{(Lfd)^2}{T}$
is given by WOLFENDEN (1927), but using different units. He has

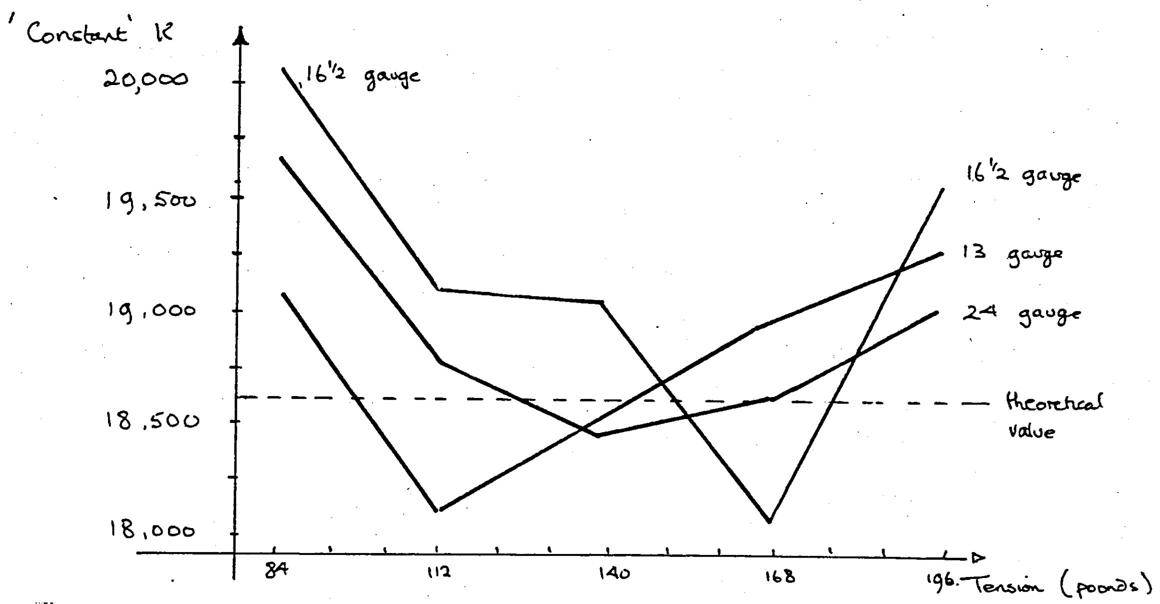
T the tension in pounds

d the string diameter in cm

L the string length in cm

and K is given as 18600.

In a paper by DARBYSHIRE (19) experiments are ddscribed in which he attempts to prove the validity of the use of constant K in pianoforte design. The results of his experiments showed that K is not a constant, but varies with tension. This requires further investigation as the variation does not seem to be due to the stiffness of the wire.



Variation of constant K with tension - 470mm string

(after Darbyshire).

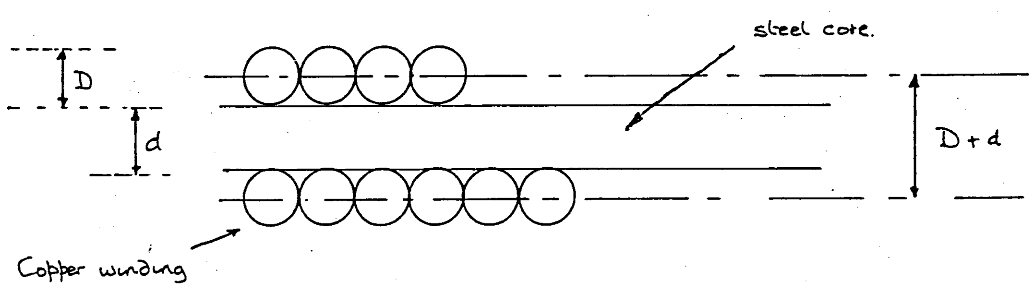
APPENDIX A2

APPLICATION OF MERSENNES LAW TO WOUND STRINGS

For wound strings, WOLFENDEN (1927) suggests the use of the constant K formulae discussed in appendix 1, modified by using K = 20 000 instead of his plain steel value of 18 600. This would only be satisfactory if the change in tension from plain to wound strings was the same in all pianos, and there was a uniform change in effective density. The effective density being due to a steel core and copper winding density being taken into account.

It should be borne in mind that cylindrical cores are not universal, a modern trend being the introduction of hexagonal cores to prevent loosening of the windings.

McFERRIN (1972) assumes that the core is cylindrical and works on the'effective mass per unit length'.



The 'effective mass per unit length' is given by

$$\mu_{eff} = \frac{\pi}{4}\left\{ \rho d^2 + \pi D (D+d) \rho_c \right\}$$

where D is the diameter of the winding

d is the diameter of the core

ρ is the core density (steel)

ρ_c is the winding density (copper)

This is not completely satisfactory however, WOLFENDEN (1927) notes that due to the winding process the finished diameter of the wound string is less than the expected sum of the components.

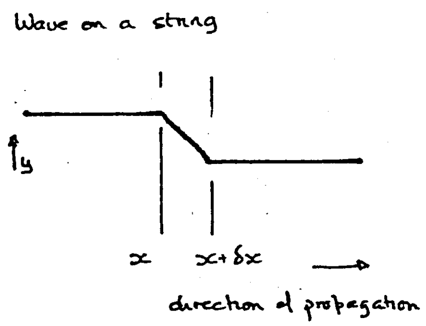
Wound strings do not have a uniform linear density, in single wound strings the winding finishes before the end of the string and only the core passes over the bridge. The situation is aggravated in the case of double wound strings in the extreme bass sections of the piano.

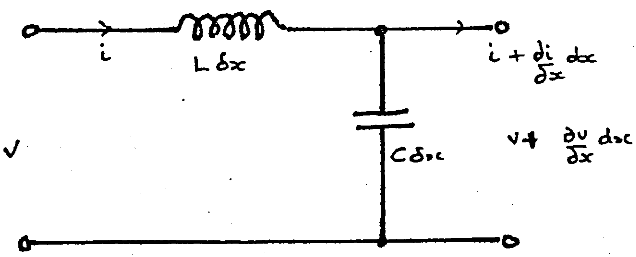
APPENDIX A3

ELECTRICAL ANALOGUE OF THE VIBRATING STRING

This important concept was first published by KOCK (1937) , and later used in a paper by EXLEY (1969). The analogy is illustrated by the following table.

STRING		TRANSMISSION LINE	
displacement	y	current	i
momentum of unit length	$\wp$	voltage	v
linear density	u	inductance per unit length	$L\partial x$
reciprocal of tension	$1/T$	capacitance per unit length	$C\partial x$
velocity of propagation	c	velocity of propagation	v_o
characteristic impedance	Z_s	characteristic impedance	Z_o
transverse velocity	$dy/\partial t$	-- -- --	$\partial i/\partial t$





Considering an element of the transmission line /string

$$\partial y/\partial x = -\tfrac{1}{T}\ \partial p/\partial t \qquad \partial i/\partial x = -C\ \partial v/\partial t$$

where   is the momentum of the string per unit length

and $\quad \partial p/\partial x = -\mu\ \partial y/\partial t \qquad \partial v/\partial x = -L\ \partial i/\partial t$

Combining these equations give the wave equation (see eqn 1.1)

$$\partial^2 y/\partial x^2 = \mu/T\ \partial^2 y/\partial t^2 \qquad \partial^2 i/\partial x^2 = LC\ \partial^2 i/\partial t^2$$

Input impedance is given by

$$Z_{in} = -j\sqrt{T\mu}\ \cot kx \qquad Z_{in} = -j\sqrt{L/C}\ \cot kx$$

Note - the transmission line is open circuit.

Variation of input impedance to the transmission line yields graphs similar to those shown on page 1.11 .

An alternative electrical / mechanical system was presented by FIRESTONE (1933) and this was used by SHARMAN (1961) to develop an alternative form of transmission line analogy. The vibration of a damped string may be represented by considering the case of a lossy transmission line.

APPENDIX ~~A4~~

BREAKING INDEX

For the fundamental mode of vibration, Mersennes Law may be written

$$f = \frac{1}{2L}\sqrt{\frac{\sigma}{\rho}}$$

where σ is the stress in the string = T / S

S is the cross section area of the string

This has been expressed by ABBOTT & SEGERMAN (1974) in terms of breaking stress and breaking frequency.

$$\text{Breaking index} = f_B L = \frac{1}{2}\sqrt{\frac{\sigma_B}{\rho}}$$

The subscript $_B$ refers to breaking.

The breaking index may be defined as the product of string length and breaking frequency for that string. The breaking index will be a constant for any string material - given the limitations of Mersennes Law outlined in section 1.7 , and Appendix 1.

Some typical figures for breaking index are tabulated below -

Material	Breaking index (m Hz)		
Piano wire	280	to	295
Nylon	aproximately		275
Phosphor bronze	145	to	165
Iron	130		165
Copper	105		120
Beryllium bronze (¾ hard)	approximately		210
Brass	100	to	175
Gut	275		295

The concept of breaking index is a useful one in instrument design.

The upper limit for the frequency of a string has been discussed by various authors, usually expressed as the maximum useable frequency divided by the breaking frequency, or the number of semitones represented by this interval.

If n is the number of semitones between maximum useable frequency and breaking frequency -

$$\text{Breaking index}_{min} > f \ L \ (\sqrt[12]{2})^n$$

This is a guide to selection of string material if the string length is known and the number of semitones below breaking pitch ass med from knowledge of similar instruments.

Typical number of semitones below breaking pitches are tabulated below -

WOLFENDEN (1916) gives data which yields		
Top string of piano	4	semitones
Lowest plain string of piano	7½	semitones
THOMAS & RHODES (1967)		
16c copper wire (harpsichord)	3	semitones
ABBOTT & SEGERMAN (1974)		
16 & 17c. (plucked instruments)	2	semitones.

APPENDIX A5

Proceedings of The Institute of Acoustics

ELECTRONIC AIDS FOR THE TEACHING OF PIANO TUNING

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Abstract

The paper outlines the procedure of 'laying the bearings' for an equal tempered scale, showing the importance of beats. A beat metronome is described based on an integrated circuit, and also a single octave practice organ which may be used for classroom demonstrations. The paper concludes by describing a device to assess the manipulative abilities of prospective students.

'Laying the bearings' and beats

The modern piano is theoretically tuned in equal temperament. The octave F3 to F4 (the octave containing middle C) is first muted with a strip of felt so that a single string in each tri-chord only will sound. When this octave has been tuned - a procedure known as 'laying the bearings' - the other strings in the octave are then tuned so that the three strings for each note sound in unison. The rest of the piano is then tuned in octaves from this original octave using felt wedges and other devices to mute strings when necessary.

The C string is first tuned to a tuning fork and then the lower F tuned to the C string. The frequency of the C string is 261.63 Hz and the theoretical frequency of the F string 174.61 Hz. Hence the F string is tuned a perfect fifth below the C and then tempered by being sharpened by 0.59 Hz ; this being the beat rate. Assuming perfect strings-

Second harmonic of 261.63 Hz	523.25 Hz
Third harmonic of 174.61 Hz	523.84 Hz
Difference frequency	0.59 Hz

The other notes in the octave are then tuned, the fifths being tempered by narrowing the interval by the appropriate amount and the fourths by widening the interval. It is possible when the beat rates are nearly correct to use checks involving sixths and thirds to assess accuracy of bearings and to correct the scale.

Electronic aids for the teaching of piano tuning

Tuning sequence and beat rates for octave F3 to F4. (A3 = 220 Hz)

C4	down to	F3	Narrow just interval by	0.59 beats per second
C4	down to	G3	Widen	0.89
G3	up to	D4	Narrow	0.67
D4	down to	A3	Widen	0.99
A3	up to	E4	Narrow	0.74
E4	down to	B3	Widen	1.12
B3	down to	F#3	Widen	0.84
F#3	up to	C#4	Narrow	0.63
C#4	down to	G#3	Widen	0.94
G#3	up to	D#4	Narrow	0.70
D#4	down to	A#3	Widen	1.05
A#3	up to	F4	Narrow	0.79

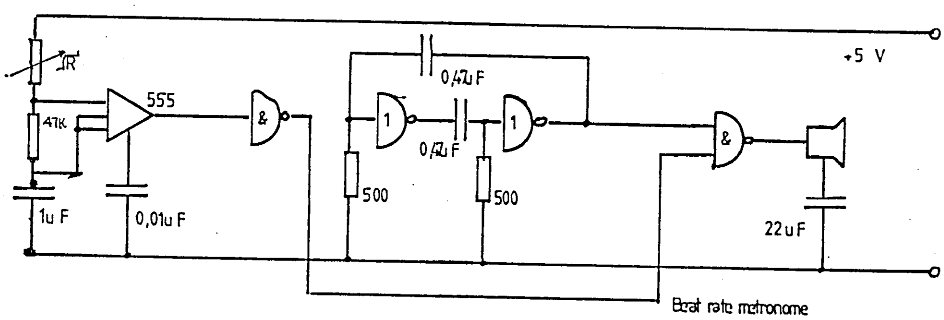
The student tuner is faced with memorising the beat rates and although this can be aided by counting against the second hand of a watch, it was decided that an electronic metronome could be built.

The beat rate metronome

The device was intended to be portable and batterypowered with accuracy maintained as long as there was an audible output. The latter was necessary as the device was to be used by blind as well as sighted students and a positive indication of low battery voltage was required. This was achieved by using a stabilised supply for the IC's which gave zero output when the battery voltage fell below 5 volts ; the timing IC was stable to well below this voltage.

The audio output is in the form of accurately timed pips; the timing being switchable as required ('tuning' beat rates or 'check' beat rates). On some prototypes the frequency of the pip tone was variable from the front panel. The metronome was calibrated using a Racal VLF counter.

Timing IC	RS 555
Nor gates (pip oscillator)	SN 7402
Nand gates	SN 7400

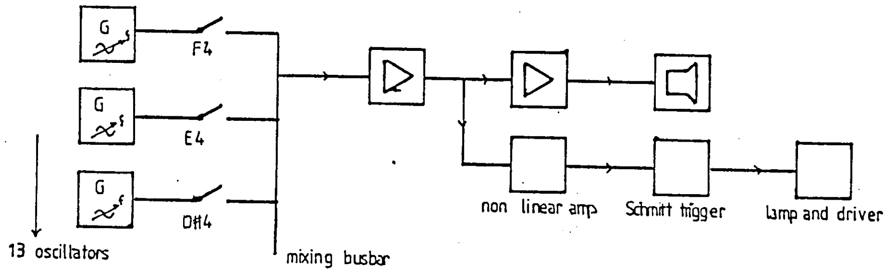


Single octave tuning practice organ

Due to the transient nature of piano tones it is difficult to demonstrate, in a classroom, beats and their use in tuning. It was decided to build a single octave practice organwhich would not only solve this problem but also be suitable as an aid for a student with difficulty in hearing or counting beats.

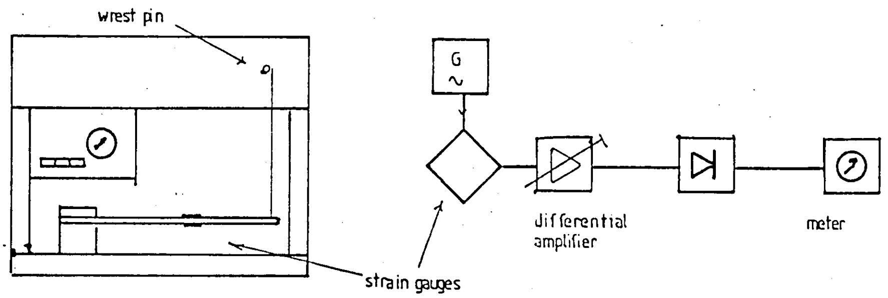
Thirteen Wien bridge oscillators covering the range F3 toF4 were built, the output being a clipped sine waveand the frequency adjusted by ten turn potentiometers. The multi-turn potentiometers were used to reduce the degree of manual dexterity required to tune the instrument.

The output was fed to a loudspeaker, and also to a non-linear amplifier and Schmitt trigger which operated a panel lamp to provide a visual indication of the beat. A secondary output was available to connect to the VLF counter to give a digital indication of the beat rate.



Manipulative assessment device

This device consists of a section of wrest plank and a wrest (tuning) pin which tensions a length of piano wire. The tension is measured using strain gauges on the supporting mild steel bar shown in the diagram. The tension is indicated by a meter, the sensitivity of which may be switched by adjusting the gain of the differential amplifier.



In use, a prospective student is asked to 'tune' the device using a standard piano tuning hammer and bring the meter pointer to a pre-determined mark. The sensitivity is re-adjusted and the procedure repeated enabling an assessment of the manipulative ability of the applicant. The sensitivities were set using the subjective judgement of a number of experienced piano tuners.

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APPENDIX OF COMPUTER AND CALCULATOR PROGRAMMES

APPENDIX C1

Calculator programme - Linear regression (T.I. 57 calculator)

Programme calculates

$$\text{Slope} = \left\{ \frac{\sum_{i=1}^{n} x_i y_i}{n} - \bar{x}\bar{y} \right\} \frac{1}{\sigma x^2} = m$$

$$\text{Intercept} = \bar{y} - m\bar{x} = c$$

Correlation coefficient = r

where $r^2 = \dfrac{m^2 \sigma x^2}{\sigma y^2}$

User instruction

STEP	PRESS	DISPLAY
1	OFF/ON LRN	00 00
2	Enter programme	
3	INV * Ct RST	
4	x_1 x⇄t y_1 R/S	1
	x_2 x⇄t y_2 R/S	2
	etc.	
5	SBR 0	Slope
	SBR 1	Intercept
	SBR 2, $\sqrt{x}$	Correlation coefficient
6	INV * Ct RST	

New data may be entered as step 4

Note, * signifies second function key

Programme

KEY	ADDRESS	CODE	COMMENT
*Σ+	00	88	
R/S	01	81	
RST	02	71	
*LBL 0	03	86 0	
RCL 5	04	33 5	
÷	05	45	
RCL 0	06	33 0	
−	07	65	
* $\bar{x}$	08	89	
X	09	55	Solves for slope
* INV $\bar{x}$	10	-89	
=	11	85	
÷	12	45	
* INV σ^2	13	-80	
=	14	85	
INV SBR	15	-61	
*LBL 1	16	86 1	
SBR 0	17	61 0	
X	18	55	
* INV $\bar{x}$	19	-89	
+/-	20	84	Solves for intercept
+	21	75	
* $\bar{x}$	22	89	
=	23	85	
INV SBR	24	-61	
*LBL 2	25	86 2	
SBR 0	26	61 0	
x^2	27	23	
÷	28	45	
* σ^2	29	80	Solves for square of correlation coefficient
X	30	55	
* INV σ^2	31	-80	
=	32	85	
INV SBR	33	-61	

APPENDIX C2

Computer programme to calculate beat rates for equal temperament assuming harmonic strings.

```
10 REM*CALLED ETBEAT ON 380Z DISC
20 REM*  J LINCOLN  1980
30 ? CHR$(12)
40 ?"EQUAL TEMPERED SCALE AND HARMONICS
50 ?"FOR F3 TO F4
60 ?"INPUT A4 FREQUENCY AND TYPE RETURN
70 ?"
80 ?"435 Hz NORMAL CONTINENTAL ,QUEENS HALL
90 ?"439    NEW PHILHARMONIC
100 ?"440    BRITISH STANDARD
110 ?"444    MEDIUM,OLD RSA
120 ?"452    KNELLER HALL
130 ?"454    OLD PHILHARMONIC
140 PRECISION 7
150 INPUT F
160 DIMA(13,8)
170 ?CHR$(12)
180 ?"A4 HAS FREQUENCY"F"Hz"
190 ?
200 ?
210 A(5,1)=F/2
220 FOR I=1 TO 13
230 IF I=5 GOTO 250
240 IF I=1 THEN A(I,1)=A(5,1)/(2^(1/3))ELSE A(I,1)=A(I-1,1)*(2^(1/12))
250 FOR J=1 TO 8
260 A (I,J) =A(I,1)*J
270 PRINT USING"######.##";A(I,J),
280 NEXT J
290 ?
300 NEXT I
310 ?
320 ?
330 E=3
340 N=1
350 ?
360 ?
370 ?"TYPE RETURN TO GIVE BEAT RATES"
380 IF E=3 GOTO 460
390 IF E=4 GOTO 520
400 IF E=5 GOTO 500
410 IF E=6 THEN E=7
420 IF E=8 GOTO 480
430 IF E=9 GOTO 540
440 ?"FOR FIFTHS
450 GOTO550
460 ?"FOR MINOR THIRDS
470 GOTO 550
480 ?"FOR MINOR SIXTHS
490 GOTO550
500 ?"FOR FOURTHS
510 GOTO 550
520 ?"FOR THIRDS
530 GOTO550
540 ?"FOR SIXTHS
550 REM*M,N ARE NEAR COINCIDENT HARMONICS
```

```
550 REM*M,N ARE NEAR COINCIDENT HARMONICS
560 INPUT ZZ
570 PRECISION 4
580 M=N+E
590 IF M>13 GOTO 870
600 REM*CONVERT NOTE NUMBER M +N TO
610 REM*ASCII IN SUBROUTINE
620 X=N
630 GOSUB 1000
640 B$=CHR$(X)+CHR$(Z)
650 X=M
660 GOSUB 1000
670 C$=CHR$(X)+CHR$(Z)
680 IF E=7 THEN G=A(N,3):D=A(M,2):GOTO740
690 IF E=5 THEN G=A(N,4):D=A(M,3):GOTO760
700 IF E=3 THEN G=A(N,6):D=A(M,5):GOTO780
710 IF E=4 THEN G=A(N,5):D=A(M,4):GOTO800
720 IF E=8 THEN G=A(N,8):D=A(M,5):GOTO820
730 IF E=9 THEN G=A(N,5):D=A(M,3):GOTO840
740 ?B$" TO "C$" FIFTH "(D-G)" BEATS NARROW
750 GOTO 850
760 ?B$" TO "C$" FOURTH "(D-G)" BEATS WIDE"
770 GOTO 850
780 ?B$" TO "C$" MINOR THIRD "(D-G)" BEATS NARROW"
790 GOTO 850
800 ?B$" TO "C$" THIRD "(D-G)" BEATS WIDE"
810 GOTO 850
820 ?B$" TO "C$" MINOR SIXTH "(D-G)" BEATS NARROW"
830 GOTO 850
840 ?B$" TO "C$" SIXTH "(D-G)" BEATS WIDE"
850 N=N+1
860 GOTO 580
870 E=E+1
880 IF E=10 GOTO 900
890 GOTO340
900 STOP
1000 REM*
1010 REM*SUBROUTINE TO CONVERT TO ASCII
1020 REM*FROM F=0,F#=2,G=3, ETC
1030 IF X<8 THEN X=X-1
1040 IF X=13 THEN X=X+1
1050 X=X/2
1060 Y=INT(X)
1070 Z=X-Y
1080 X=Y
1090 IF X>1 THEN X=X+63
1100 IFX<=1 THEN X=X+70
1110 REM * VAL Z    32=SPACE, 70=#
1120 Z=Z*70
1130 IF Z=0 THEN Z=32
1140 RETURN
```

APPENDIX C3

Calculator programme for partial frequencies of a stiff string fundamental (f_1) tuned to equal temperament

T I 57 Calculator.

Programme calculates f_0 from f_1 and β^2, using

$$f_0 = f_1 / \sqrt{1 + \beta^2}$$

This result is stored and used to calculate partials using

$$\therefore \quad f_n = n f_0 \sqrt{1 + \beta^2 n^2}$$

β^2 is input as data, having been previously calculated from string dimensions using

$$\beta^2 = 4.15 \times 10^6 \frac{d^2}{L^4} \frac{1}{f_1^2}$$

User instruction

STEP	PRESS	DISPLAY
1	OFF/ON LRN	00 00
2	Enter programme	
3	LRN RST	
4	Numerical value of β^2	
5	STO 1, INV EE	
6	Numerical value of f_1	
7	R/S	Flashes 1, displays f
8	R/S	Flashes 2 displays f
	etc.	
9	* INV Ct, RST	
	New data may be entered as step 4	

Note, * signifies second function key

Programme

KEY	ADDRESS	CODE	COMMENT
-	00	45	
(	01	43	
1	02	01	
+	03	75	
RCL 1	04	33 1	Calculates f_0 from f_1
)	05	44	
x	06	24	
=	07	85	
STO 2	08	32 2	
*LBL 1	09	86 1	
1	10	01	
SUM 0	11	34 0	
* INV FIX	12	-48	
RCL 0	13	33 0	Displays partial number as integer
* Pause	14	36	
x	15	23	
X	16	55	
RCL 1	17	33 1	
=	18	85	
+	19	75	
1	20	01	
=	21	85	
x	22	24	
X	23	55	
RCL 0	24	33 0	
X	25	55	
RCL 2	26	33 2	
=	27	85	
* FIX 2	28	48 2	
R/S	29	81	Displays partial frequency to 2D places
GTO 1	30	51 1	

APPENDIX C3

Computer programme to predict the temperament for a piano with stiff strings when tuned to beat rates for fourths and fifths traditionally used (those for harmonic strings)

```
10 REM*
20 REM*CALLED STIFF ON 380Z DISC
30 REM*
40 REM*PROGRAMMED  J LINCOLN 1980
50 REM* HDE 9123
60 REM*
70 REM*PROGRAMME PREDICTS TEMPERAMENT
80 REM*FOR A PIANO IF THE PARTIALS ARE
90 REM*CALCULATED TAKING THE STRING
100 REM*STIFFNESS INTO ACCOUNT
110 REM*
120 REM*THE BEAT RATES ARE THOSE
130 REM*CALCULATED IF EQUAL TEMPERAMENT
140 REM*AND HARMONIC STRINGS ARE ASSUMED
150 REM*
160 REM*ONLY BEAT RATES FOR FOURTHS AND
170 REM*FIFTHS USED
180 REM*
190 REM*PROGRAMME SUBSCRIPT 1 REFERS F3
200 REM*                    2        F#3
210 REM*                    3        G3
220 REM*       ETC
230 REM*
240 REM*TUNING RANGE CONSIDERED IS FROM
250 REM*F3 TO F4
260 REM*
270 REM*TO CALCULATE BEAT RATES FOR FOURTHS AND FIFTHS
280 REM*HARMONIC STRINGS
290 REM* C1 REFERS FIFTH F3 TO C4
300 REM* Q1 REFERS FOURTH F3 TO A#3
310 REM*    ETC
320 ?"HARMONIC BEAT RATES"
330 F=440
340 REM*ARRAY TO HOLD HARMONICS OF NOTES F3 TO F4
350 DIM D(13,4)
360 D(5,1)=F/2
370 FOR I=1 TO 13
380 IF I=5 GOTO 400
390 IF I=1 THEN D(I,1)=D(5,1)/(2^(1/3))ELSE D(I,1)=D(I-1,1)*(2^(1/12))
400 FOR J=1 TO 4
410 D (I,J) =D(I,1)*J
420 NEXT J
430 NEXT I
440 E=7
450 DIM C(6)
460 FOR N=1 TO 6
470 M=N+E
480 G=D(N,3):D=D(M,2)
490 C(N)=D-G
500 ?"C"N,C(N)
510 NEXT N
```

```
520 ?"
530 E=5
540 DIM Q(8)
550 FOR N=1 TO 8
560 M=N+E
570 G=D(N,4):D=D(M,3)
580 Q(N)=D-G
590 ?"Q"N,Q(N)
600 NEXT N
610 REM*FOR PIANO OTHER THAN THAT USED
620 REM*IN EXPERIMENTS ADD
630 REM*GOTO 720
640 DIM B(13)
650 B(1)=3.3295726344E-04:B(2)=3.3087128598E-04
660 B(3)=2.7014712845E-04:B(4)=2.6697524236E-04
670 B(5)=2.9170532475E-04:B(6)=3.0708219249E-04
680 B(7)=3.3836727861E-04:B(8)=3.7033264634E-04
690 B(9)=4.1121501335E-04:B(10)=4.3717414732E-04
700 B(11)=4.8239430119E-04:B(12)=5.3461224411E-04
710 B(13)=5.7836828221E-04
720 GOTO 880
730 ?CHR$(12)
740 REM*TO CALCULATE STIFFNESS FACTOR B
750 ?"WHEN NOTE NAME IS FOLLOWED BY ?
760 ?"INPUT STRING LENGTH AND DIAMETER
770 ?"ALL DIMENSIONS IN METRES
780 DIM B(13)
790 DATA "F","F#","G","G#","A","A#","B","C","C#","D","D#","E","F"
800 S=2^(1/12)
810 FOR N=-4 TO 8
820 READ N$:?N$;:INPUT L,D
830 F=220 * (S^N)
840 GOSUB 2020
850 B(N+5)=B
860 ?"B"(N+5),B(N+5)
870 NEXT N
880 DIM A(13,5)
890 REM*ARRAY TO HOLD PARTIAL FREQUENCIES OF
900 REM*NOTES F3 TO F4 (STIFF STRINGS)
910 REM*FUNDAMENTAL OF A2 SET TO 220 Hz
920 A(5,1)=220
930 A(5,0)=A(5,1)/SQR(1+B(5))
940 FOR J=2 TO 4
950 A(5,J)=A(5,0)*J*SQR(1+B(5)*J^2)
960 NEXT J
970 REM*FIFTH UP A TO E
980 M=5 :GOSUB 2050
990 REM*FOURTH DOWN E TO B
1000 M=12:GOSUB 2150
1010 REM*FOURTH DOWN B TO F#
1020 M=7: GOSUB 2150
1030 REM*FIFTH UP F# TO C#
1040 M=2 :GOSUB 2050
1050 REM*FOURTH DOWN C# TO G#
1060 M=9 :GOSUB 2150
1070 REM*FIFTH UP G# TO D#
1080 M=4:GOSUB 2050
1090 REM*FOURTH DOWN D# TO A#
1100 M=11:GOSUB 2150
1110 REM*FIFTH UP A# TO F
1120 M=6:GOSUB 2050
1130 REM*FOURTH DOWN F TO C
1140 M=13:GOSUB 2150
```

```
1150 REM*FOURTH DOWN C TO G
1160 M=8:GOSUB 2150
1170 REM*FIFTH UP G TO D
1180 M=3:GOSUB2050
1190 REM*OCTAVE DOWN F TO F
1200 M=13
1210 N=M-12
1220 A(N,2)=A(M,1)
1230 GOSUB 2080
1240 FORI=1 TO 13
1250 PRINT "
1260 FOR J=0 TO 4
1270 PRINT USING "######.##";A(I,J),
1280 NEXT J
1290 NEXT I
1300 PRINT"
1310 PRINT"
1315 ?"PIANO TEMPERAMENT DEVIATION FROM EQUAL TEMPERAMENT"
1320 DIM E(13)
1330 FOR K=1 TO 13
1340 R=A(K,1)/D(K,1)
1350 E(K)=1200*LOG(R)/LOG(2)
1360 PRINT K,E(K)
1370 NEXT K
2000 REM*****SUBROUTINES**********
2010 STOP
2020 B=4.15*10^6*D^2/(L^4*F^2)
2030 RETURN
2040 STOP
2050 REM*FIFTH UP , M UP TO N
2060 N=M+7
2070 A(N,2)=A(M,3)+C(M)
2080 A(N,0)=A(N,2)/(2*SQR(1+4*B(N)))
2090 A(N,1)=A(N,0)*SQR(1+B(N))
2100 FOR J=3 TO 4
2110 A(N,J)=A(N,0)*J*SQR(1+J^2*B(N))
2120 NEXT J
2130 RETURN
2140 STOP
2150 REM*FOURTH DOWN ,M DOWN TO N
2160 N=M-5
2170 A(N,4)=A(M,3)-Q(N)
2180 A(N,0)=A(N,4)/(4*SQR(1+16*B(M)))
2190 FOR J=1 TO 3
2200 A(N,J)=A(N,0)*J*SQR(1+J^2*B(M))
2210 NEXT J
2220 RETURN
2230 STOP

READY:
```